Mathematics of The Elegant Universe

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Overview:

From time to time, teachers should put themselves in their students' shoes. "If I were in this class, would I be interested and engaged? As a teenager, would I be excited about a sine wave, an isosceles triangle, or an improper fraction?" Math for its own sake can seem pointless to adolescents. It has been my experience that students are more motivated to learn mathematics when it is presented within a relevant and challenging context. If I want my dog to take a pill, I hide it in a piece of hamburger. With students, the juicy morsel of an unusual context can foster in them the desire to learn. As an example, the study of trigonometry can seem like a dry exercise in mental gymnastics until it is applied to interesting problems. It is the application of abstract theory to concrete reality that shows students the value of our intellectual apparatus. The more interesting the application, the stronger the concomitant allure. I think this idea is well-captured by Brian Greene, author of The Elegant Universe, in an interview with National Public Radio, aired on May 29, 2005. "I am distressed when I meet students who approach science and math with drudgery. I know it doesn't have to be that way. But when science is presented as a collection of facts that need to memorized, when math is taught as a series of abstract calculations without revealing its power to unravel the mysteries of the universe, it can all seem pointless and boring." I agree with him completely.

In my reading of The Elegant Universe, I found literally dozens of things which fascinated and perplexed me, many of which were beyond the scope of a high school math class. But I also found many applications which were within the intellectual reach of my students. I decided to focus on three areas which I felt were both within the ability of my students and within the scope of my curriculum. I open the unit with some questions which I hope will catch my students' attention. Is it true that if I were to travel on a spaceship at 99.9% of the speed of light, I could return to earth to see the year 2500, and meet my great, great, great, great, great, great grandchildren? Why do I see a rainbow shimmering on the surface of a CD? How can I measure the wavelength of the red light coming out of my laser pointer? How can I use my laser pointer to determine the spacing of the grooves on a CD? Is space flat or curved? What experiments could be performed to determine the nature of the space we live in? All of these questions can be discussed using mathematics accessible to students in an Elementary Functions class. So the three topics of my unit are time dilation, diffraction grating of visible light, and non-Euclidean geometry.

Topic 1: Time Dilation

Rationale: In solving this problem, students will employ formulas for distance, rate, and time, they will use the Pythagorean Theorem, and they will simplify fractions within a radical. (PSSA Assessment Anchors 2.1.8, 2.2.8, 2.8.8, and 2.10.8) I chose this topic because time dilation is a phenomenon which is likely to capture the imagination of many students. Most people do not realize that our experience of time is relative to our frame of reference. They assume that, if synchronized watches at rest with respect to each other are in perfect working order, then they will stay synchronized, even if one watch undergoes an acceleration with respect to the other watch. This is, in fact, not true. When I am in motion with respect to another person, I experience time at a different rate than they do. We will disagree as to the amount of time which has elapsed. And incredibly, we will age at different rates. The difference in elapsed times in not a mere trick of our watches, but an actual difference in the amount of time experienced during the event. These differences are too small to notice at relatively low velocities such as we experience in our life on Earth, but become quite noticeable at speeds approaching the speed of light. Experiments have confirmed that time dilation is a fact of reality. I am anxious to see if my students find this as amazing, mind-boggling, and downright shocking as I do. I anticipate that some students will be stimulated to read more about time dilation on their own after this brief introduction to the concept. I also feel that the use of the Pythagorean Theorem in relating the elapsed time recorded by the two observers is a valuable augment to the standard Pythagorean triangle which the students have worked with in the past. The symbolic manipulation required to solve for the elapsed time of the observer on the platform is both challenging and worthwhile.

Objective: The students will be able to derive the Lorenz transformation for time dilation using a light clock traveling on a train passing a platform. They will then use this formula to make predictions about elapsed time during various high-speed trips.

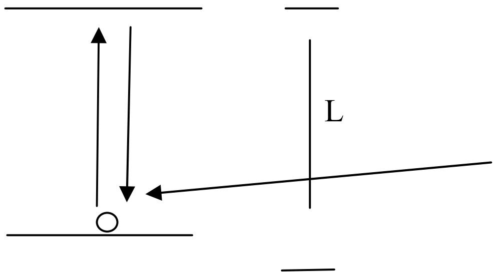
Development: I would engage students with the following statement, which sounds like something from science fiction: If I climb into a spaceship and travel at near the speed of light for a few days, when I return, several centuries will have passed on earth and all of my friends will be dead of old age, while I will still be young. How could they have aged while I didn't? The situation seems absurd, but Einstein insists that this is really what would happen. To understand the paradox of the time traveler, we must first consider what time means.

In the 1600's, Newton developed something known as classical physics (the standard high school physics) based on several assumptions, one of which was that space and time are absolute. "Absolute, true, and mathematical time, of itself and from its own nature, flows equably without relation to anything external, and by another name is called "duration"; relative, apparent, and common time is some sensible and external (whether accurate or unequable) measure of duration by means of motion, which is commonly used instead of true time, such as an hour, a day, a month, a year." So true or absolute time flows smoothly and invisibly, as sand through an hourglass, and though it is invisible, we try to measure it using regularly repeating, periodic phenomena. The duration of a single period of one of these phenomena becomes our time currency, used to compare the duration of events. Our choice of currency could be a single rotation of the earth on its axis (a day), a rotation of the earth around the sun (a year), or the swing of a very reliable pendulum (a second). "My trip to Texas took five days" compares the duration of my trip to the duration of one rotation of the earth about its axis. Newton never really said anything about what time itself is, but only that, whatever it is, it flows equably, which means in a steady, non-fluctuating, and uniform manner. He says I will experience time at the same rate whether I am sitting on the beach reading a book, driving a car, flying in a rocket, or falling off a cliff. Newton said that how we are moving does not affect the flow of time, and since he never saw a situation where time failed to flow smoothly, this seemed like a reasonable assumption. Whenever we observe motion at low velocities, we observe a smooth change in spatial location and a smooth change in time, and the time **seems** to flow equably. As we shall see, something very strange happens at very high velocities, something no scientist anticipated until Einstein.

Newton says later, "It may be said that there is no such thing as an equable motion whereby time may be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is not liable to any change." Newton realized that there might be no such thing as perfectly periodic phenomena. For example, friction causes motion to slow, so any timekeeper based on motion with friction would tend to slow down and lose accuracy. But to repeat, Newton says that the flow of true time is steady and constant.

Newton's concept of absolute time began to unravel when it was discovered that the speed of light in a vacuum is always 'c' (approximately 186,000 miles per second), no matter how the person doing the measuring is moving. For example, if a beam of light is coming towards two observers, and one observer is traveling towards the beam, while the other is traveling away from the beam, they will both measure the light as c. This is surprising. From experience with objects at slow speeds, we know that if a ball is coming towards us, it hits us harder if we are running towards the ball, and softer if we are running away from the ball. Our motion relative to the ball affects the velocity with which we are hit. Two football players collide with much greater speed when they are running towards each other, and much less speed when they are both running in the same direction. It seems like common sense. (I will demonstrate it in class by tossing an eraser to a student walking towards me, then to a student walking away from me.) So how could two observers measure the exact same speed of light if one observer moves towards the beam and the other moves away from the beam? They just do. And since they do, that's just how it is. This leads to some weird predictions, and we will examine one of them.

Since we can use any periodic function to measure time, I propose we use a photon of light bouncing back and forth between two mirrors which are spaced exactly L feet apart. A "tick" is a unit of time defined as the amount of time it takes for the photon to move from the bottom mirror to the top, reflect, and then return to the bottom mirror.

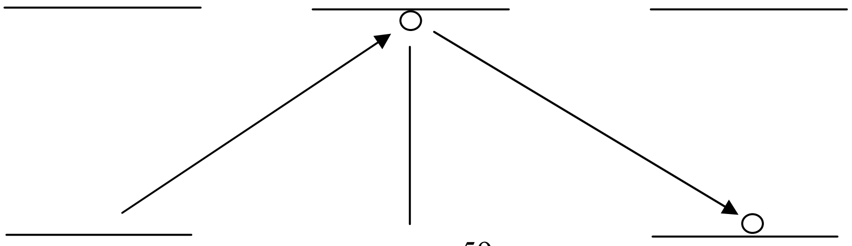


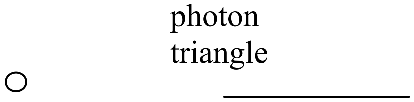
From Jacob's perspective, the photon travels back and forth a distance of 2L every "tick".

The clock is traveling along smoothly on a train, with Jacob riding on the train with the clock and Maria standing on a platform as the train passes by. Jacob watches as the photon goes up and down in front of him, and calculates the amount of time for one "tick" as the distance divided by the velocity, or:

$$T_J = \frac{2L}{C}$$

However, Maria measures a different amount of time for one "tick" on Jacob's clock. As the train goes by, at velocity V, she sees the photon of Jacob's clock moving in a saw-tooth pattern, first moving diagonally up and then diagonally down.





From her point of view, the photon covers a greater distance than 2L. The hypotenuse of the photon triangle is the distance light travels in a half-tick, or C * $T_m/2$. One leg of the photon triangle is L, and the other is the distance the train moves in a half-tick, or V * $T_m/2$. Using the Pythagorean Theorem, we set the sum of the squares of the legs of the triangle equal to the square of the hypotenuse, and solve for T_m.

$$\left(\frac{CT_m}{2}\right)^2 = L^2 + \left(\frac{VT_m}{2}\right)^2$$

$$\frac{C^2 T_m{}^2}{4} - \frac{V^2 T_m{}^2}{4} = L^2$$

$$\frac{T_m{}^2}{4}\left(C^2 - V^2\right) = L^2$$

$$T_m{}^2 = \frac{4L^2}{C^2 - V^2}$$

$$T_m = \frac{2L}{\sqrt{C^2 - V^2}}$$

$$T_m = \frac{2L}{\sqrt{C^2 - \frac{V^2 C^2}{C^2}}}$$

$$T_m = \frac{2L}{C\sqrt{1 - \frac{V^2}{C^2}}}$$

$$T_m = \frac{\frac{2L}{C}}{\sqrt{1-\frac{V^2}{C^2}}}$$

At this point, we notice that the numerator is exactly the elapsed time according to Jacob, so we can replace the numerator with T_J, yielding:

$$T_m = \frac{T_J}{\sqrt{1-\frac{V^2}{C^2}}}$$

The denominator is always less than one!!!

$$T_m \geq T_J$$

The result shows that the elapsed time according to Maria is the elapsed time according to Jacob divided by a number which is less that one. So Maria will say each "tick" took more time than Jacob, and so more time will pass for her than for Jacob. At this point, the class will plug in actual values for V, the velocity of the train, and calculate T_m for each value. If V = .9993 C, then $T_m = 26.7\ T_J$. So each second for Jacob will seem like 26.7 seconds for Maria. Maria would actually be aging faster than Jacob, and the flow of time would be faster for her than for Jacob. If Jacob rode a train with V = .999995 C, each second for Jacob would seem like 316 seconds for Maria, or about 5.25 minutes. At this velocity, Jacob would say one day passed on the train, and Maria would say 21 years passed. If Jacob traveled for a couple of weeks at this velocity, and then returned to earth, he would find Maria dead of old age, while he would still be young.

I will ask the class about what our transformation says about elapsed time for an object traveling at light speed. The formula for V = C results in a zero denominator, which is undefined. In theory, an object traveling at light speed would experience no elapsed time. It would be ageless. This is philosophically interesting to me.

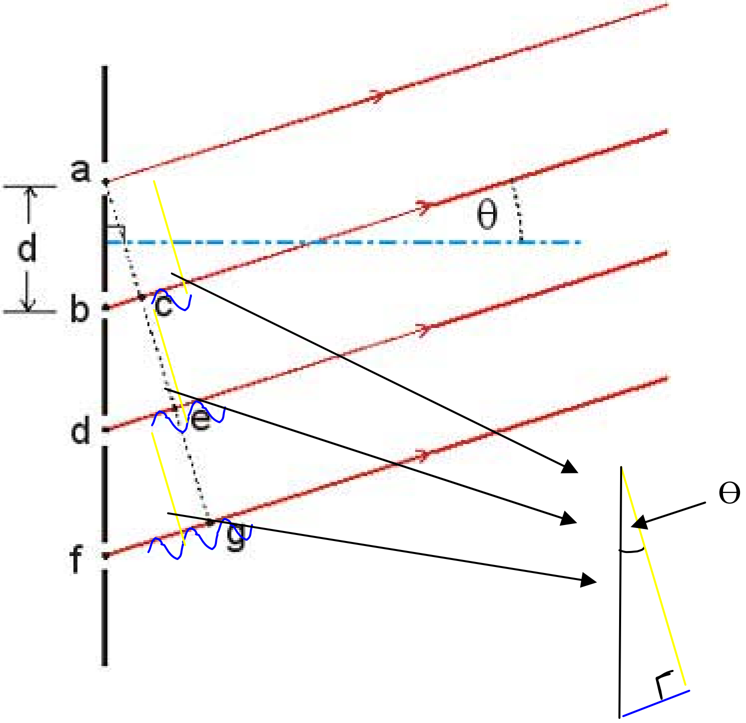
This lesson is planned for two class periods. The first period will be spent discussing the concept of time and setting up the light clock. The second period will be spent solving the photon triangle for the elapsed time and plugging in various velocities to determine the amount of time dilation in each case.

Topic 2: Diffraction Grating

Rationale: Electrodynamics allows students to apply their understanding of sinusoidal wave transformations in an interesting way. The solutions to these problems require students to apply their understanding of parallel lines, alternate interior angles, complementary angles, similar triangles, and phase shifts. (PSSA Assessment Anchors 2.3.8, 2.10.8, and 2.11.8) I chose this topic because students in Elementary Functions study graphing transformations extensively. They learn to manipulate sine and cosine waves through shifting and stretching both vertically and horizontally, and also reflecting across various axes. These manipulations are performed on theoretical waves. The textbook provides examples where students get to apply this analysis to the motion of a buoy floating on the ocean, but optics provides a beautiful and even deeper application of waves in reality. The concept of in-phase and out-of-phase will strengthen the student's ability to visualize the waves as they are manipulated through a series of transformations. I also think the right triangle formed by the diffraction grating provides a challenging application of the sine function itself. The angle ϴ appears in two different locations in the picture. Identifying these two angles as equal is a review of several critical geometry concepts. Students must see ϴ as complementary to an angle ά, and also involved in a linear triple with this same angle ά and a right angle. Logically, this makes the two angles ϴ equal. But this is far from obvious when the students are shown the diagram without the ϴ marked in both spots. I would require the students to form this connection.

Objective: Students will be able to calculate the first and second order maxima and minima for a diffraction grating of known spacing.

Development: The lesson will begin with a brief explanation of constructive and destructive interference. Light is a form of electromagnetic radiation. It can be thought of as traveling through space in the shape of a sine wave. Violet light has a wavelength of about 400 nm, and red has a wavelength of about 700 nm. When two waves travel together, they interfere with each other. Constructive interference occurs when two waves travel in phase, which means their peaks and valleys are aligned. If two waves of equal strength travel in phase, the result is a wave with double intensity. Destructive interference occurs when two waves travel out of phase, which means their peaks and valleys are opposed. If two waves of equal strength travel out of phase, they cancel each other out and no light is observed.



http://www.saburchill.com/physics/chapters2/00151.html (Modified from the original.)

Imagine a light source with wavelength λ encountering a barrier with tiny holes of separation D. Beams, traveling as sine waves, pass through each hole and then continue on until they hit a wall. We can ask ourselves, where on the wall will the light beams interfere constructively, and where will they interfere destructively. The light beams are in phase as they pass through the holes, and some of the light will continue straight forward, in phase, until it hits the wall. Here we expect to see a bright spot, since the waves are traveling in phase. Where else would we expect constructive interference? Where else would the waves travel in phase? I would ask the students for suggestions. Eventually, we would determine that they would travel in phase when the difference in path length for any two adjacent beams is exactly λ. We could model the beams as parallel lines, with an angle of deflection from the normal as Θ degrees. Students will work in pairs to try to determine Θ_1 so that the path length differences are λ. This occurs when $\lambda = D * \sin(\Theta_1)$. So we expect to find a bright spot at Θ_1, called the first order maxima. The light would also travel in phase if the path differences are 2 λ. This occurs when $2\lambda = D * \sin(\Theta_2)$. So we expect another bright spot at Θ_2, called the second order maxima. We can also calculate the angles of destructive interference where the waves travel out of phase. Students will work in pairs to calculate Ω_1 and Ω_2, the first and second order minima where we expect to find dark regions on the wall.

After this, students will calculate actual values for Θ_1 and Θ_2 and Ω_1 and Ω_2 for a violet beam of light of wavelength 440 nm, passing through a diffraction grating with 6000 grooves per cm.

After this, students will shine my laser pointer through the same diffraction grating, and measure the angles for Θ_1 and Θ_2. They will use these angles to calculate λ, the wavelength of the light emitted from my laser pointer.

After this, students will shine my laser pointer at a CD, measure Θ_1 and Θ_2, and then calculate the spacing on the grooves on my CD. Calculating the wavelength of the light given the maxima angles involves the sine function, while calculating the maxima given the wavelength of the light involves the inverse sine function. This will reinforce the connection between these two functions.

The spacing on a disc is approximately 1600 nm. The pickup light used to read the CD has a wavelength of 500 nm, making the spiral groove in a CD over 3 miles long. The secondary maxima, Θ_2, is used by the CD reader to guide the pickup light. Sensors receiving the secondary maxima adjust the direction of the light to keep it moving along the groove. This application of diffraction grating is unusual.

Another interesting fact is that the tracks in the CD are cut, leaving a pit depth of 125 nm. I would ask the students to speculate on why this depth is chosen by engineers. The answer is simple. Since this is 25% of the wavelength of the pickup light, light reflected from the pit travels 250 nm more than light reflected from the surface, making it out of phase with light reflected from the surface.

This lesson is designed to be delivered over three class periods. In the first period, students will work in the lab using internet wave simulators to learn about wave propagation. www.falstad.com has excellent double slit simulators where students can adjust the frequency of the light and the width and spacing of the openings. The destructive interference is especially evident there.

Topic 3: The Geometry of Space

Rationale: Students in high school study Euclidean geometry, and most don't even realize that other types of geometry are possible. This topic will allow students to explore two other types of geometry, and consider methods to test which type of geometry actually describes our universe. (PSSA Assessment Anchors 2.2.8, 2.3.8, 2.4.8, and 2.9.8)

Objective: Students will discuss Euclid's 5th postulate, and using an equivalent form called Playfair's postulate, students will develop alternative postulates. Students will measure the angles of a triangle in spherical geometry, and use Girard's theorem to estimate the area of the circle.

Development: On the first day of the unit, students will construct great circles using rubber bands around a globe. Students will use great circles to find the shortest distance between any two points on the globe. Students will see that the shortest distance between two cities often takes them along an unexpected path. For example, the great circle from Chicago to London passes much further to the North than most students expect. Students will create triangles using rubber bands to form the sides, and then measure the three interior angles of the triangle to show that the sum is greater than 180 degrees. They will also see that great circles always intersect, so if great circles represent lines, there are no parallel lines in spherical geometry.

The second day of the unit will be spent on a proof of Girard's Theorem. Consider a triangle T on a sphere. We will be deriving a formula for the area of T. The key to understanding the derivation is the configuration of the three great circles on the sphere. There is no difficulty understanding what you see there. What might cause problems is what the configuration looks like on the other side of the sphere.

We will label the vertices of T by **R**, **G**, and **B**, and the corresponding angles of T by r, g, and b. The letters stand for red, green, and blue, and, for example, the vertex **R** is the vertex of T where T is opposite a red triangle. The angles at **R** in the triangle T and in the red triangle are opposite angles and therefore are equal. Their value will be denoted by r. In fact **R** is the vertex of two congruent lunes, one of which consists of the red triangle and a gray triangle, and the other of which contains the black triangle and another red triangle. We will refer to these two lunes as the red lunes. We will denote by L_r' the red lune which does not contain T, and by L_r the red lune which does contain T. In exactly the same way we see that **G** is the vertex of two congruent, green lunes --- L_g which contains T, and L_g' which does not contain T, and the vertex **B** is the vertex of two congruent, blue lunes --- L_b which contains T, and L_b' which does not contain T.

If you rotate the sphere you will also see a triangle that looks pretty much the same as T. This is the antipodal triangle T'. Its vertices are **R'**, **G'**, and **B'**, which are the points antipodal to **R**, **G**, and **B** respectively. Since T and T' are images of each other under the antipodal map, which is an isometry, they have the same area.

It is important to understand the situation of each pair of like colored lunes. Concentrate on the two blue lunes, L_b and L_b'. Notice the triangle T is part of the lune L_b and the triangle T', which is antipodal to T, is part of L_b'. Examination of the other pairs of lunes reveals that the lunes L_g and L_r also contain T, while L_g' and L_r' contain T'.

To sum up, the six lunes L_r, L_r', L_g, L_g', L_b, and L_b', have the following properties:

- The triangle T is contained in each of the three lunes L_r, L_g, L_b, and in no others.
- The antipodal triangle T' is contained in each of the three lunes L_r', L_g', L_b', and in no others.
- Every point of the sphere which is not in T or T' is contained in precisely one of the lunes.

Understanding the proof of Girard's Theorem comes down to understanding the configuration of the triangle and the six lunes, and verifying the three bulleted points. By far the best way to visualize the six lunes is by physical experimentation with an actual sphere. Get a beach ball about 8 to 12 inches in diameter. Draw a triangle T on it. Then carefully extend each side of the triangle to a complete great circle. It will be noticed that these great circles intersect on the other side of the sphere and form another triangle T' which is the antipodal image of T. Thus T' is congruent to T and consequently has the same area.

Suppose that the three angles of T are **R**, **G**, and **B**. At each of these vertices there are two lunes of the appropriate angle that meet. One of them contains T. Call this lune L_r, L_g, and L_b as the case may be. Denote the other lune, which does not contain T by L_r', L_g', and L_b'. Hatch the two lunes L_r and L_r' with a distinctive color or marking (such as little circles). Hatch the lunes L_g and L_g' with a different color or marking, and use yet a third for the lunes L_b and L_b'.

Now by examining the beach ball you will be able to verify the three bulleted points.

We can sum up the bulleted points by saying that the six lunes cover the entire sphere with the points in T and T' covered two additional times. Therefore when we add up the areas of the lunes we have

$$\begin{aligned}\text{area}(L_r) + \text{area}(L_g) + \text{area}(L_b) + \text{area}(L_r') + \text{area}(L_g') + \text{area}(L_b')\\ = \text{area(sphere)} + 2\,\text{area}(T) + 2\,\text{area}(T').\end{aligned}$$

Into this equation we substitute the formulas for the area of a lune, and the surface area of a sphere of radius R. Finally, using the fact that T and T' have the same area, we get

$$2R^2r + 2R^2g + 2R^2b + 2R^2r + 2R^2g + 2R^2b = 4\pi R^2 + 4\,\text{area}(T).$$

Next, solving for the area of T, and collecting terms this becomes

$$\begin{aligned}\text{area}(T) &= \frac{1}{4}\left(2R^2\left(2r + 2g + 2b\right) - 4\pi R^2\right)\\ &= R^2\left(r + g + b\right) - \pi R^2\\ &= R^2\left(r + g + b - \pi\right).\end{aligned}$$

This last formula is called *Girard's formula*, and the result of the formula is called *Girard's Theorem*.

We get an interesting variant if we solve for the sum of the angles:

$$r + g + b = \pi + \frac{1}{R^2}\,\text{area}(T).$$

Both formulas are interesting. The first emphasizes the area of the spherical triangle, and the second emphasizes the sum of the angles of the spherical triangle. For comparison with planar geometry, the second is especially interesting because it says precisely how much the sum of the angles of a spherical triangle exceeds two right angles, the sum of the angles for a planar triangle. That the difference involves the area of the sphere is a remarkable departure from what we would expect from our knowledge of plane geometry.

The final day of this topic will involve a discussion about how we could tell if our universe was curved. Students will discuss whether beams of light should serve as lines in the geometry of our universe. There will be a brief history of attempts to measure the interior angles of triangles formed by light, with stars at the vertices. Students will be asked to speculate why triangles of such scale are needed to detect any deviation from 180 degrees. Hyperbolic geometry will also be mentioned as an alternative to Euclidean geometry. Students will discuss how Playfair's Postulate would be altered to form hyperbolic geometry.

The PSSA Assessment Anchors for Mathematics are:
2.1 Numbers, Number Systems and Number Relationships
2.2 Computation and Estimation
2.3 Measurement and Estimation
2.4 Mathematical Reasoning and Connections
2.5 Mathematical Problem Solving and Communication
2.6 Statistics and Data Analysis
2.7 Probability and Predictions
2.8 Algebra and Functions
2.9 Geometry
2.10 Trigonometry
2.11 Concepts of Calculus

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NOVA Program on *The Elegant Universe* by Brian Greene