Fractals

Roseann Y. Casciato

Taylor Allderdice High School

Contents of Curriculum Unit

Overview

This unit of study on fractals will be incorporated into my 10^{th} grade Gifted (Center of Advanced Studies – CAS) Algebra 2 class. The purpose of this unit is to introduce students to the topic of fractals. In teaching a gifted class, it is expected that I add enrichment into the existing curriculum. It is my job to introduce students to topics other than ones that are present in their textbook. I should be able to expose them to mathematical topics that are present in the real-world so they can see the relevance in them. I think this unit would fit nicely in our Algebra 2 textbook at the end of Chapter 5 on Irrational and Complex Numbers.

Each lesson should take two days to complete so a total of 12 days are needed. Of course, if this is too much time to take out of your original curriculum, you can do as much or as little as you see fit. Also, once you do one or two lessons, you may like to assign some of the other lessons as homework for students to do outside of the classroom. This is still a nice enrichment activity and you do not have to use all of your class time to complete it. In order to introduce a unit on fractals, other than explaining the history of them, I think it is best to start out with a very basic lesson. For my students to do any assignments, they need to first become aware of new terminology. The first lesson will introduce the basic concepts of geometric iterations. Students will practice applying geometric iteration rules to generate a sequence of figures then predict what future iterations would evolve. I will introduce terminology such as seed, orbit, iterate, period, and cycle to them. The next lesson will introduce students to fractals that are created by following a specific set of rules. These fractals are called deterministic fractals because their outcome is determined by consecutive applications of the rules. At this point, students are still not introduced formally to fractals but hopefully they are starting to have an idea of what a fractal is. The third lesson is where I will introduce self-similarity. Well known fractals such as the Sierpinski triangle, the Koch curve and the Cantor middle-thirds set will be introduced. In the fourth lesson students will have the opportunity to play the chaos game. I think my students will enjoy playing the chaos game. I will have them play in class and then we will go to the computer lab to play online. The fifth lesson will have students investigating the Sierpinski triangle and the different ways to generate this fractal. The last lesson in this unit will deal with fractal dimension. Students will have to use their knowledge of exponents and logarithms in this lesson. Students will see not only how beautiful fractals are, but also how they have important applications in our lives.

Rationale

Why take a seminar called Fractals and Chaos? What is so important about fractals and chaos that I would want to learn in order to inform my students about the topics? Fractals are a "hot" topic recently. Throughout this seminar I was given the opportunity to examine how fractals and chaos are present all around us in nature. Some of the mathematics that I teach on a daily basis goes back as far as 300 B.C. with Euclid and his work with geometry. My students are always asking why are we learning a certain mathematical lesson or when will we ever use this again. They feel as though the mathematics they are learning is from the past and they are just going through the motions of solving a problem. It would be nice for me to share with my class a new and more recent mathematical topic.

"Historically, F. Hausdorff (1919) introduced the term fractal dimension in the sense of noninteger dimension and with the given definition the age of fractality has begun. Consequently, a set that can be assigned a fractal dimension is called a fractal set" (Sandau). Other people over the years have researched fractals. For example, "Benoit Mandelbrot was largely responsible for the present interest in fractal geometry. He showed how fractals can occur in many different places in both mathematics and elsewhere in nature" (O'Connor, J.J. and Robertson, E. F.). Benoit was given one of Gaston Julia's papers to read. The paper was *Mémoire sur l'iteration des fonctions rationelles* which made Gaston very famous in the mathematical world. Gaston's paper dealt with the iterations of a rational function f. Benoit worked at IBM's Watson Research Center laboratories in New York State. With the help of computers, Benoit was able to take the information that Gaston wrote about to show us just how beautiful some of these fractals truly are. For Benoit to do this, he had to develop not only new mathematical ideas, but also he had "to develop some of the first computer programs to print graphics" (O'Connor, J. J. and Robertson, E. F.). The use of computers and computer graphics made it possible for people to see what fractals actually looked like, as opposed to only envisioning them in their minds. This is a direct connection to the technology that permeates the modern educational system. Mandelbrot also observed that many objects such as clouds and coastlines could be explained using fractal geometry as opposed to Euclidean geometry.

Benoit Mandelbrot who is famous for discovering the Mandelbrot Set developed the term "fractal" in 1975. "Mandelbrot coined the word "fractal" (from the Latin word "fractus", meaning fractured, broken) to label objects, shapes or behaviors that have similar properties (self-similarity) at all levels of magnification or across all times, and which dimension, being greater than one but smaller than two, cannot be expressed as an integer" (Martinez). His work is obviously a continuation of many mathematicians' work from previous years. However, since that the term fractal was just introduced in the seventies, you can see how exciting it would be to be able to incorporate it into my classroom.

I have been teaching Algebra 2 for the past 14 years and the textbooks we use have approximately 3 pages dealing with fractals and mention only the most basic concepts. I am piloting a new Algebra 2 textbook this year and it is hard to believe but they still do not have a section devoted to fractals. They only mention Benoit Mandelbrot and the use of complex numbers in fractal geometry. One would think by now, you would see more emphasis on the topic of fractals.

The advantage of teaching this unit to my CAS Algebra 2 students is that they have already taken geometry. A lot of the work in this unit is conceptual so it may be necessary to review some mathematical terms or equations such as the Pythagorean Theorem, the 45º - 45º - 90º triangle, similar polygons, perimeter and area. Logarithmic functions are not introduced until chapter 8 so I will have to give a brief explanation about them before we attempt the last lesson in this unit. Because this lesson on fractals does not necessarily fit into an exact unit of study in the CAS Algebra 2 curriculum, I am treating it as an extension to the gifted curriculum.

"The term "mentally gifted" includes a person who has an IQ of 130 or higher, when multiple criteria as set forth in Department Guidelines indicate gifted ability. Determination of gifted ability will not be based on IQ score alone. A person with an IQ score lower than 130 may be admitted to gifted programs when other educational criteria in the profile of the person strongly indicate gifted ability. Determination of mentally gifted shall include an assessment by a certified school psychologist" (CAS Centers for Advanced Study Teacher Handbook 7). "The CAS program has been designed to meet the needs of the gifted students for individualized, accelerated and enriched learning. All CAS classes emphasize an inquiry approach to learning, problem solving techniques and the higher cognitive skills of analysis, synthesis and evaluation" (CAS Centers for Advanced Study Teacher Handbook 3). The CAS class sizes are limited to 10 – 18 students which helps allow for more individualized attention. In addition to the regular course work for each class, students are required to complete a Long Term Project (LTP). The LTP should be on a topic that interests the students because they have to research it in dept, work a minimum of 30 hours outside of the classroom and then do a presentation to their class.

As a teacher it is my responsibility to learn about new developments in the teaching world, particularly mathematics. Being a member of the National Council of Teachers of Mathematics, I get a monthly subscription to the magazine, The Mathematics Teacher. Over the past few years I've noticed fractals are becoming an important topic discussed in the magazine. Prior to this seminar I never attended any classes nor taught my students about fractals. Due to what I learned in this seminar, I will incorporate fractals in my future lesson plans. Even though I have a specific curriculum to cover in my existing classes, I think it is important to inform students of new topics that are evolving in the mathematical world. Hopefully my students will enjoy doing the mathematics necessary to develop a fractal as well as seeing that fractals are beautiful geometric shapes that can be generated by simple rules.

Why would someone reading this curriculum unit want to try the lessons in their classroom? Actually there are a few reasons why I want to teach this unit on fractals and hopefully you will agree that it is worth teaching to your students. I would love for my students to learn concepts such as:

- seeing that fractals are a more recent mathematical topic,
- making connections between mathematics and the real world,
- expanding their inductive, deductive, explicit and recursive reasoning skills.

There is more to this unit than just drawing a few examples of fractals. This unit is not like something from a traditional textbook. Our students will be given the opportunity to learn about a topic that otherwise they may never be exposed to. I cannot begin to explain how important I think it is while teaching to make connections between mathematics and the real world. I believe that once students are exposed to this topic, they will want to pursue it at greater length. I am expecting this unit to reach some students that may not have been too engaged in mathematics before. This is truly something totally different than what they have been taught in class before. If I am able to spark an interest in my students I will be thrilled.

When I was growing up I thoroughly enjoyed learning mathematics. I especially liked exploring patterns and finding relationships among things. I would like my students to enjoy this aspect of mathematics as well. I do not want them to just sit in my classroom, watch me work out problems and then practice the same concepts at home. I want my students to see there is a world of mathematics out there. I want them to explore and investigate topics. It is important for students to have the opportunity to research mathematical topics and to understand why things occur in mathematics. Students should also be able to think logically and make inferences about whatever mathematical topic they are researching.

"Traditionally, high school algebra instruction has emphasized manipulating symbols to solve equations and simplify expressions. In contrast to the traditional role of school algebra as following procedures for manipulating symbols, *Principles and Standards for School Mathematics* recommends that students in grades 9-12 will

> generalize patterns using explicitly defined and recursively defined functions;…use symbolic algebra to represent and explain mathematical relationships;…[and] use symbolic expressions, including iterative and recursive forms to represent relationships arising from various contexts" (Lannin 216-217).

This is crucial because this shows that today we are expecting our students to reason mathematically recursively and explicitly. But why is this so important? What is the difference between recursive and explicit? When students begin examining patterns, they are reasoning recursively. “Recursive reasoning uses an established mathematical relationship between a previous term or terms in a sequence” (Lannin 217). For explicit-preferred tasks, students tend to develop explicit rules because the recursive relationship is not evident or because the recursive relationship varies in such a way that it cannot be easily determined” (Lannin 219). All of this discussion on reasoning is very significant when dealing with fractals. Students will be using inductive, deductive, explicit, and recursive reasoning when exploring fractals.

Not only is the way students think and perceive things important, but I would also like them to make connections between mathematics and the real world. Being a mathematics teacher is not only doing problems from a textbook. It is necessary to give thorough explanations of the topics you are teaching and make connections to real world situations. This lesson gives me the opportunity to show students that fractals are not just a mathematical term, but they do in fact exist in the real world. When I begin this lesson, the first day will be devoted to discussing fractals in general. Throughout this discussion, I am hoping students will be able to give some examples of what they think a fractal is rather than my telling them. This will challenge students to use their own deductive reasoning skills. I will be certain to discuss fractals in nature in particular. We can talk about leaves, clouds, coastlines and seashells to name a few. It is also beneficial for students to know that a wide variety of people deal with fractals on a daily basis, such as doctors, art historians and computer programmers. I am looking forward to teaching this lesson because this is a major opportunity for me to truly connect mathematics to the real world. For all of the times that I am asked when are we ever going to use this, I will be happy to be able to give excellent examples as a response.

Objectives

“In high school, students should build on their prior knowledge, learning more-varied and more-sophisticated problem-solving techniques. They should increase their abilities to visualize, describe, and analyze situations in mathematical terms. They need to learn to use a wide range of explicitly and recursively defined functions to model the world around them. Moreover, their understanding of the properties of those functions will give them insights into the phenomena being modeled” (NCTM 288). One of my main goals in this lesson is to have students uncover the area of mathematics known as fractal geometry. When I teach, I always try to incorporate real-life application problems in my lesson so students understand they are not just “doing math”. Students need to make the connection that mathematics exists all around us. This lesson works extremely well in that students will learn about a new topic and see how it relates to them everyday. What is nice is that it is not just mathematics they are learning about. They will observe that many objects around them are fractal in appearance.

My students have grown up in a technological world and they are very computer savvy and enjoy electronics. It will be very nice for them to see how beautiful some of these fractals are by using a computer, instead of trying to envision what they look like. It is important for me to explain to my students that fractals are present in more places other than mathematics. “Fractals arise in medicine: Cancerous tumors, human lungs, and vascular systems are all examples of fractals. Art historians use fractals to date early Chinese paintings. Seismologists use fractals to study the fissures caused by earthquakes. Computer programmers use fractal techniques to encode large sets of data efficiently. Fractals even occur in Broadway plays (such as Tom Sheppard's *Arcadia*) and in films (such as *Jurassic Park*), where they are used to create extraterrestrial planet-scapes and other special effects" (Choate et al. Fractals XII). Obviously there will be enough mathematics involved in the different lessons for me to make connections of fractals to other areas of mathematics. I will incorporate graphing calculators as well as computers in the lesson as well.

Some of the connections in mathematics that students will find involve similar figures, ratios of similarity, geometric transformations, geometric sequences, exponential growth and decay, Pythagorean Theorem, special right triangles (45º - 45º - 90º triangle), fractions, perimeter, area, similarity, congruence, exponential growth and decay, similarity, measurement, probability, coordinates, logic, problem solving, Pascal's Triangle, volume, Euclidean and non-Euclidean geometry and measurement. As you can see, there is a wide range of topics that students will be exposed to. I believe this will be an important unit of study because students will truly make the connection of mathematics to the real world.

It is imperative that I incorporate the Pennsylvania State Standards (Appendix 1) into my lesson plans. During this unit, the seven mathematics standards will be addressed in this curriculum unit. Students will be expected to use number systems, compute and solve practical problems, apply the concepts of patterns, formulate and solve problems and communicate the mathematical processes used, understand and apply basic concepts of algebra and geometry to solve theoretical and practical problems, evaluate, infer and draw appropriate conclusions from charts, tables and graphs, showing the relationships between data and real-world situations, and lastly, make decisions and predictions based upon the collection and interpretation of statistical data and the application of probability.

Strategies

This unit of study will be comprised of lecture, independent and group work. I believe it is important for students to become active participants in the classroom. I do not like to lecture at students on daily basis. I try to the best of my ability to have students examine the mathematics and make findings on their own as well actually teaching to them. A lot of people feel lecturing to students is the only way learning takes place. Having students sitting in single file rows not talking to each other but rather listening to the teacher lecture is not necessarily the best way to have all students learn. I believe it is vital for all students to take an active part in the classroom in order to enhance their learning process. I truly believe that "hands on" activities give students the opportunity to derive material and appreciate what it is they are working on. I hope that through incorporating different teaching styles, I will be able to reach more students and have them enjoy this unit of study on fractals. For further detail on strategies, see the section below on classroom activities.

Classroom Activities

Because this is probably the first exposure that students have with fractals, I would start this unit with students doing some research on the topic. I would like them to have some prior knowledge of fractals in order to help stimulate a classroom discussion. Once we spend a period discussing issues such as the creation of fractals and where they exist in the mathematical and real world, we could then move into the first lesson.

Lesson 1

This first lesson is important because I have to make sure students are exposed to terminology they will need in order to be successful in this unit. I will begin this lesson as a whole classroom group discussion. Students will learn about iterations, a seed, an iteration rule and orbit. Because fractals are created by iterations, students will repeat a specific process multiple times to create a sequence of figures. The sequence of shapes that exist are called the orbit of the geometric iteration. I want students to be able to tell me what shape will evolve after multiple iterations are performed.

The first two examples that I will work through with my class deal with the shrinking iteration rule. I will have an overhead transparency for students to follow and they will have the same transparency as a worksheet (Appendix 2). It is expected that students take notes and continue the orbit on their worksheet. The first example that students will be exposed to is having a seed as a square with sides of length 1. They will shrink the figure so that each side is half as long. After numerous iterations, students will see that the squares shapes are becoming a single point. Another example similar to the square could be to use a stick figure for the seed. When performing iterations, it is important for students to know that decreasing the shape by 2 is the same as shrinking by ½.

Another iteration rule that I can show the class is the rotating iteration rule. Following the same format as above with the shrinking iteration rule, this time I would start with a seed of an ellipse with a face inside of it, and then rotate it 90º clockwise (Appendix 3). Students will hopefully notice that the cycle repeats itself every fourth iteration; this orbit is a cycle of period 4. I would also want students to see a seed which is a filled circle. The resulting orbit is just a filled circle, nothing changes. This orbit is called a fixed orbit because it remains the same.

After we finish the above examples, students will work in pairs on the next part of the lesson. I will have a worksheet (Appendix 4) prepared and have students try the shrinking and rotating iteration rules. The seed they will begin with are two vertical lines (||). They will sketch the next three pictures in the orbit shrinking the previous figure so that each side is one-half as long and one-half as wide. They will then each have to create their own seed and sketch the next four pictures. I will ask them to describe what happens in each of the orbits and what they think the orbit will approach. Hopefully students will see that after four iterations there are still two lines, but in the long term, they will eventually approach a single point.

I will have students use the rotating iteration rule sketching the first four pictures in the orbit of a square, ellipse and a half circle (Appendix 5). Students will see the square has an orbit that is fixed, the ellipse has an orbit of cycle of period 2 and the half circle has an orbit of cycle of period 4.

The last example students will work on together deals with a well-known fractal called the Sierpinski triangle. Students will start with a solid triangle (Appendix 6). The iteration rule is to remove a triangle from the middle of the given triangle so that three congruent triangles remain. I will show the seed and first iteration, and then have students draw the next two figures of the orbit. Once they finish their drawings, they will have to write a few sentences describing their observations and complete a table to determine the nth iteration.

For homework (Appendix 7) students will start with a seed of a straight line and the iteration rule is: "remove the middle third of any segment, leaving behind the endponts" (Choate et al., Fractals 16). The resulting fractal is another famous fractal known as the Cantor middle-thirds set. Students will be expected to write a short paragraph explaining the fractal they developed and complete a table to determine the nth iteration.

Lesson 2

In this next lesson students will examine fractals that are formed using a precise set of rules. These fractals are called deterministic fractals because their "fate is determined by successive applications of the rules" (Choate et al., Fractals 19). Because this lesson will involve dividing images into smaller pieces, similar to the original and congruent to each other, students need to be familiar with geometric shapes such as congruent, equilateral and right triangles, as well as regular polygons and exponents. Since this lesson is being incorporated into my Algebra 2 class, I will have to take some time in class to refresh the memory of my student's geometric concepts.

In this lesson I will have the seed and rule on the overhead transparency, the student will have the same transparency as the worksheet (Appendix 8), and I will have the class discuss what the first removal would result in. The different seeds that I will start with are an equilateral triangle and its interior (to create Sierpinski's triangle), a square and its interior (to create the Fractal Plus) and another square with its interior (to create the Fractal H). This should take a whole class period with students interacting and sketching the next few iterations at their desks. I will also have students volunteer to come to the overhead and sketch what they think should be the next iteration. The two fractals that are formed from using a square and its interior, I will be sure to give students graph paper to use in order for them to sketch a more accurate fractal and to save time drawing. I am sure the question will arise as to how much of the graph paper students should use when beginning with their seed. Both of these fractals start with a square whose side is length 1. It is then broken into nine squares whose sides have length 1/3. This should lead into a nice class discussion because they are going to be expected to divide the fractal correctly on their graph paper. In addition to teaching about fractals, this lesson in particular gives a nice segue into symmetry. Students should be able to determine lines of symmetry and which fractal, Fractal Plus or Fractal H has more lines of symmetry than the other ones. For homework I will assign the Sierpinski Carpet. The Sierpinski Carpet is formed with the iteration rule: "Begin with a square whose side has length 1. Then divide the square into nine equal-sized squares whose sides have length 1/3. Remove the middle squares, leaving eight smaller squares behind" (Choate et al., Fractals 29). I will have students draw the next few figures in the orbit and complete a table for each iteration with the total number of squares remaining and the side length of each square (Appendix 9). It is expected that they can determine what the shape would ultimately look like if they continued this process.

LESSON 3

The third lesson in this unit deals with self-similarity. "An object is said to be self-similar if it looks "roughly" the same on any scale. Fractals are a particularly interesting class of self-similar objects" (Weisstein, Math World). "Self-similarity in a figure means that the overall shape of the figure is the same as the shape of a smaller part inside the figure, which in turn has the same shape as an even smaller part, which in turn has the same shape as an even smaller part, and so forth" (Choate et al., Fractals 38). This lesson can begin with students observing different fractals that will be placed on the overhead projector. This will be a good time to discuss what is meant by the terminology of contraction factors and magnification factors. For example, if students were to use a magnifying glass, they would see the piece of the fractal actually double. This is what is known as a magnification factor of 2. "Incidentally, when we say "magnify by a factor of 2," we mean that the length and width of the figure is doubled. The area is thus multiplied by 4" (Choate et al., Fractals 39). If students were seeing that a figure was being reduced, this is what is known as a contraction factor. I would begin this lesson with an overhead transparency of Sierpinski triangle (Appendix 10). A whole group class discussion is necessary at this point to make certain students understand first of all that what they are looking at is in fact the Sierpinski triangle and we would start a discussion on all of the magnifications and contractions that they see. "Given our initial triangle was a seed, we removed the middle triangle; this left us with three smaller triangles, each of which was exactly one-half the size of the original. At the second stage of the construction, we found nine smaller triangles that were one-fourth the size of the original. And at the nth stage, we found 3^n triangles that were $\frac{1}{2}^n$ times the size of the original" (Choate et al., Fractals 40). Depending on the fractal that you are examining determines the magnification factor. For example, if you happen to observe a fractal that is one-third the size of the original figure, the magnification factor of for that fractal is 3. At this point in the lesson, I will have students working in pairs on two investigations. They will analyze The Sierpinski Carpet and a Fern Fractal. The fern fractal will give the students exposure to real-world fractals and how they exist in nature. Students will be given a picture of the Sierpinski Carpet (Appendix 11) and they will have to find each self-similar piece that has a magnification factor of 3 and complete a table that will ask for the magnification factor and the number of self-similar pieces. They will also be expected to find a piece of the fractal that demonstrates the fractal's self-similarity at a smaller scale. They will have to write the total number of pieces at this scale. In looking at the fern fractal I would first want to know what the students have to say about the self-similarity of this fractal. Also, can the students take this topic further and find other items in the real-world that display self-similarity to some level. To further the students' understanding of fractals, self-similarity, and magnification, I will have students construct The Cantor Middle-Thirds Set and have them answer questions dealing with it (Appendix 12).

Lesson 4

In the fourth lesson, students will have the opportunity to play The Chaos Game. The Chaos Game is a nice introduction into the topics of chaos and fractals. This exercise "is an excellent problem-solving challenge in recognizing patterns" (Choate et al., Fractals 68). The process of this game is random, and most people who play this game think the results will be random, but that is untrue. Students will discover the shape of the Sierpinski triangle once they are finished playing and compare their overhead transparencies. Students will work with a partner when playing this game. They will first play it manually then they will have the opportunity to go to the computer lab and play a version online (http://math.bu.edu/DYSYS/applets/fractalina.html). The reason I want students working in pairs is because one student can roll the die while the other student records the number. When getting started with this lesson, it is essential that each pair of students have a die, a ruler, overhead transparencies and markers to mark their findings. I would suggest giving every partner the same handout (Appendix 13) as a transparency for them to record their findings. After all of the games are finished being played, you can place each of the transparencies on top of each other on the overhead and hopefully the students will realize the image of the Sierpinski triangle is being formed. They should recognize that the middle of the triangle is not being filled in. I will also show students the results of the game after 300 iterations (Appendix 14) and then after 150,000 iterations (Appendix 15).

To begin playing the game without the use of a computer, each of the partners will have a transparency with three points at the vertices of a triangle. The three points will be labeled "1, 2", "3, 4" and "5, 6". The numbers are such because depending on what number is displayed when you roll your die, determines which vertices you use during the iterations. Before any iteration can begin, each student needs to have a starting point in the interior of the triangle. This starting point is called a seed. We can call the seed z_0. The iteration rule is that you first roll the die to determine a vertex. Depending on which number comes up, you will have to move to a new point z_1 which is halfway between z_0 and the target vertex. The students will continue rolling the die for approximately 30 times continuing this half way process. Obviously one question to ask your class is what figure will become apparent? It is important to start a discussion and have students explain why they think the pattern that is evolving actually does.

The following day in class the students will have the opportunity to go to the computer lab and play the Chaos Game online. For homework each student will receive a worksheet (Appendix 16) with Sierpinski triangles on it with z_0 in the center of the middle white triangle then have a shaded triangle and ask what the student would have to roll to land on the shaded triangle if they only were permitted to have two iterations then ask the same question if they did three iterations. I will ask specific questions (Appendix 17) that I expect the students will be able to answer pertaining to the lesson. For example, why does the chaos game which is played with the three vertices materialize into the Sierpinski triangle? Is it necessary to start with a seed that lays inside the triangle? What happens if you start with a seed that lays outside the triangle? I would want the students to show an example to support your findings.

Lesson 5

When I stated earlier in this paper that fractals are not a very big part of our curriculum, the one fractal that seems to be mentioned in our textbooks is the Sierpinski triangle. Due to this fact, I would like this next lesson to deal with the Sierpinski triangle and any connections that students can make. One very interesting connection they will see deals with Pascal's triangle. I will give the students Pascal's triangle with the numbers, and then have the shape of a triangle made from small circles in the position of the numbers from Pascal's triangle (Appendix 18). Students will then shade in the circles which correspond to odd numbers and leave the even number entries empty. Students will be expected to explain what the result will be. Students should also be able to explain whether to shade a circle based on the shading of the circles directly above it and also explain any relationship they see to the addition of odd and even numbers.

Lesson 6

The last lesson in this unit is Fractal Dimension. I do not think a unit in fractals would be complete if fractal dimension was not discussed. Fractal dimension is seen in many real-world applications. People of different occupations have been able to measure the fractal dimension of a curve such as a cell membrane, coastline and/or landscape edge. Doctors use an X-ray image to determine if a tumor is benign or malignant. A malignant tumor is seen as a problematic shape while a benign tumor is smoother. The radiologist is able to differentiate between the two through dimension because the smaller the dimension, the more likely the tumor is benign. Geologists are able to predict when and where earthquakes will appear. Art historians use fractal dimensions to estimate paintings. This is a perfect time to explain to your students different occupations where fractal dimension is present.

Because logarithms are used in dealing with fractal dimension, it would help if you incorporated this lesson after you studied logarithms. What is fractal dimension? To begin with, do your students even understand the term dimension? That is, a line is one dimension, a plane is two dimensions, and a cube is an example of three dimensions. Why is this? What makes these shapes one, two or three dimensions? Have students join you in a discussion as to why these dimensions are one, two or three. Students will come to learn that fractal dimension deals with the roughness or thickness of the fractal. “The dimension is simply the exponent of the number of self-similar pieces with magnification factor **N** into which the figure may be broken” (Devaney, Fractal Dimension). Therefore, the formula for the fractal dimension of an object is $n = m^d$, where n = number of self-similar pieces, m = magnification factor and d = dimension. “Fractal dimension is a measure of how "complicated" a self-similar figure is. In a rough sense, it measures "how many points" lie in a given set” (Devaney, Fractal Dimension). The fractal dimension that exists is not a whole number, but a fraction.

Students will sketch the fractal for the Sierpinski triangle, The Cantor middle-thirds set and The Koch curve (Appendix 19). They will have to determine the number of self-similar pieces that the fractal can be divided into, the magnification factor that corresponds to the number of self-similar pieces they choose then the fractal dimension of the fractal. To help you further explain fractal dimension, see (Appendix 20) which deals with ordering fractals by dimension. This worksheet will have eight different pictures of fractals and students will be expected to determine the fractal dimension of each one. They will also have to calculate the exact dimension of each fractal. Hopefully, your students will be able to determine why they were correct or incorrect with their original guesses.

At this point it is necessary to explain how the dimension of the Sierpinski triangle is found and to incorporate logarithms. One way to incorporate this into your lesson is by showing the students a picture of the Sierpinski triangle; they should be able to tell you that it can be broken into three self-similar pieces each of which may be magnified by a factor of 2 to produce the entire triangle. Using the definition of fractal dimension stated above, students should be able to write the equation $3 = 2^d$. Taking the log of both sides gives $\log 3 = \log(2^d)$, using properties of logarithms, $\log 3 = d \log 2$, then dividing both sides by $\log 2$, you generate $d = \frac{\log 3}{\log 2} \approx 1.585$. Therefore, the Sierpinski triangle has the dimension approximately 1.585.

Works-Cited

Choate, Jonathan, Devaney, Robert L., and Foster, Alice. Fractals – A Tool Kit of Dynamic Activities. California: Key Curriculum Press, 1999.

Devaney, Robert L.. "Chaos in the Classroom." April 2, 1995. Boston University. < http://math.bu.edu/DYSYS/chaos-game/chaos-game.html>.

Devaney, Robert L.. "Fractal Dimension." April 2, 1995. Boston University. <http://math.bu.edu/DYSYS/chaos-game/node6.html>.

Lannin, John K. "Developing Mathematical Power by Using Explicit and Recursive Reasoning." Mathematics Teacher Volume 98, Number 4 November 2004.

Martinez, Juan Luis. "Benoit Mandelbrot." Feb. 19, 2003. Third Apex to Fractovia. <http://www.fractovia.org/art/people/mandelbrot.html>.

National Council of Teachers of Mathematics. Principles and Standards for School Mathematics. Virginia: NCTM, 2000.

O'Connor, J.J. and Robertson, E. F. "Benoit Mandelbrot." July 1999. <http://wwwhistory.mcs.standrews.ac.uk/Mathematicians/Mandelbrot.html>.

Pittsburgh Public Schools Committee Members. CAS Centers for Advanced

Study Teacher Handbook. Pittsburgh: Pittsburgh Public Schools, March 2005.

Sandau, Konrad. “Fractals in Nature and How to Measure Them.” Feb. 12, 2000. Santa Fe Institute. <http://www.fbmn.fh-darmstadt.de/~sandau/biofractals/abstract_sfi.html>.

Weisstein, Eric W.. "Self-Similarity." From *MathWorld*--A Wolfram Web Resource. <http://mathworld.wolfram.com/Self-Similarity.html>.

Appendix 1

Mathematics Content Standards

1. All students use numbers, number systems, and equivalent forms (including numbers, words, objects and graphics) to represent theoretical and practical situations.

2. All students compute, measure and estimate to solve theoretical and practical problems, using appropriate tools, including modern technology such as calculators and computers.

3. All students apply the concepts of patterns, functions and relations to solve theoretical and practical problems.

4. All students formulate and solve problems and communicate the mathematical processes used and the reasons for using them.

5. All students understand and apply basic concepts of algebra, geometry, probability and statistics to solve theoretical and practical problems.

6. All students evaluate, infer and draw appropriate conclusions from charts, tables and graphs, showing the relationships between data and real-world situations.

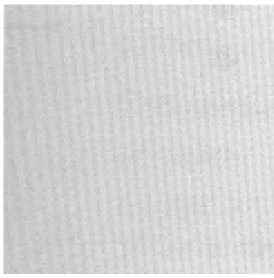
7. All students make decisions and predictions based upon the collection, organization, analysis and interpretation of statistical data and the application of probability.

Appendix 2

Shrinking Iteration Rules

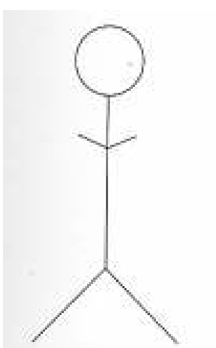
Seed: A square with sides of length 1.

Rule: Shrink the figure so that each side is half as long. Draw the next three figures in this orbit.



Seed: Stick figure.

Rule: Shrink the figure so that each shape formed is half as big as the previous shape. Draw the next three figures in this orbit.

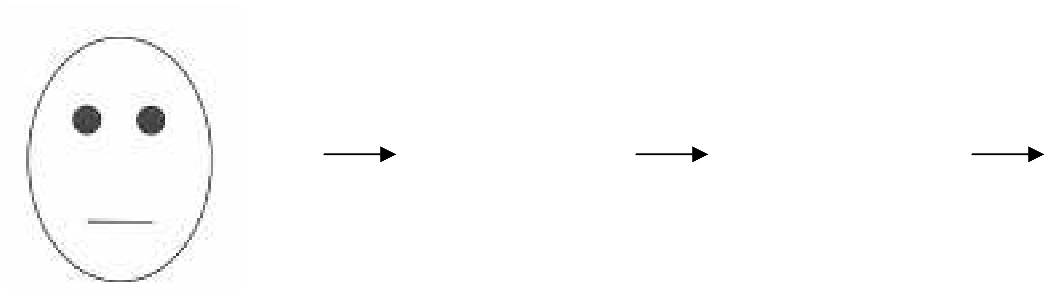


Appendix 3

Rotating Iteration Rule

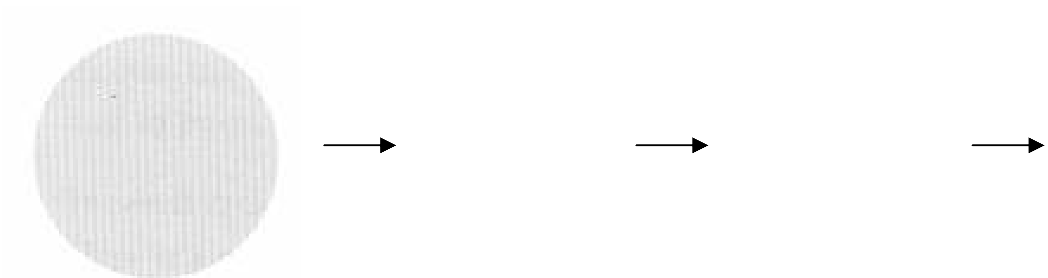
Seed: Ellipse with face inside of it.

Rule: Rotate the ellipse in the plane $90°$ clockwise. Draw the next three figures in this orbit.



Seed: Filled circle.

Rule: Rotate the ellipse in the plane $90°$ clockwise. Draw the next three figures in this orbit.

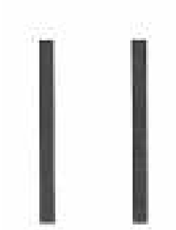


Appendix 4

Shrinking Iteration Rule

Seed: Two vertical lines.

Rule: Shrink the previous figure so that each side is one-half as long as wide. Draw the next three figures in this orbit.



Does this orbit approach one point or two points? Why?

Create a seed and follow the same rule as above.

Does this orbit approach one point or two points? Why?

Appendix 5

Rotating Iteration Rule

Seed: Square, ellipse and half circle.

Rule: Rotate the figure in the plane 90° clockwise. Draw the first four figures in this orbit.

Square

→ → →

Ellipse

→ → →

Half circle

→ → →

Appendix 6

The Sierpinski triangle

Seed: Solid triangle.

Rule: Remove a triangle from the middle of the given triangle so that three congruent triangles remain. Draw the next three figures in this orbit.

 → → →

After many steps you will have removed the middles of all triangles. The resulting fractal is called the Sierpinski triangle. Write a short paragraph describing this fractal.

Complete the table below in order to determine the nth iteration.

	Original triangle (the seed)	First Iteration	Second Iteration	Third Iteration	nth Iteration
Total number of triangles remaining	1	3			
Side length of each triangle	1				

Appendix 7

The Cantor middle-thirds set

Seed: Straight line.

Rule: Remove the middle third of any segment leaving behind the endpoints.

Draw the first few figures in this orbit.

After many steps you will have removed the middles of line segments. The resulting fractal is called the Cantor middle-thirds set. Write a short paragraph describing this fractal.

Complete the table below in order to determine the nth iteration.

	Original segment (the seed)	First Iteration	Second Iteration	Third Iteration	nth Iteration
Total number of segments remaining	1	2			
Length of each segment	1				

Appendix 8

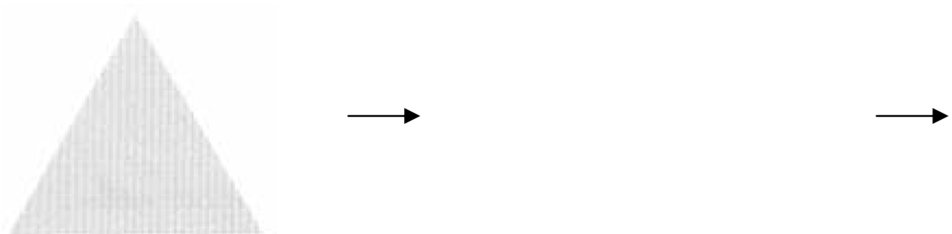
Generating Fractals by Removals

The Sierpinski triangle
Seed: An equilateral triangle and its interior.

Rule: Remove a triangle from the middle of the original triangle so that:

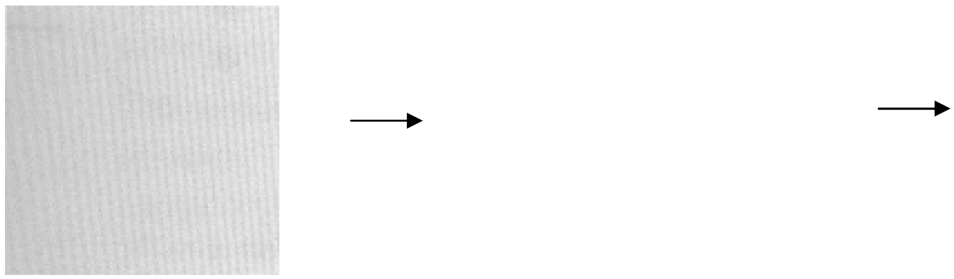
- The sides are one-half the length of the original.
- Three congruent triangles remain.

Draw the next two figures in this orbit.



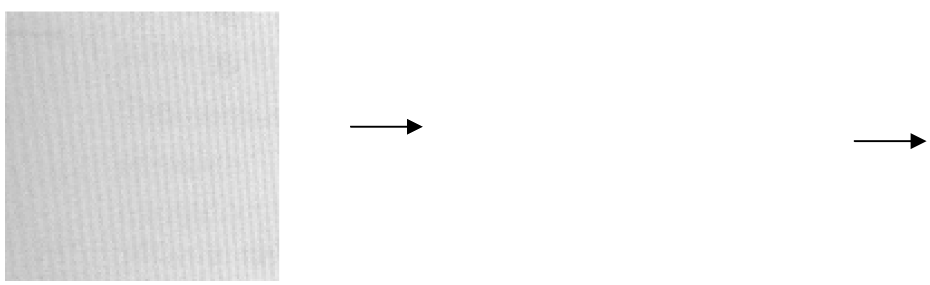
Fractal Plus
Seed: Square whose side has length 1.

Rule: Divide the square into nine squares whose sides have length $\frac{1}{3}$. Then remove the four corner squares. Draw the next two figures in this orbit.



Fractal H
Seed: Square whose side has length 1.

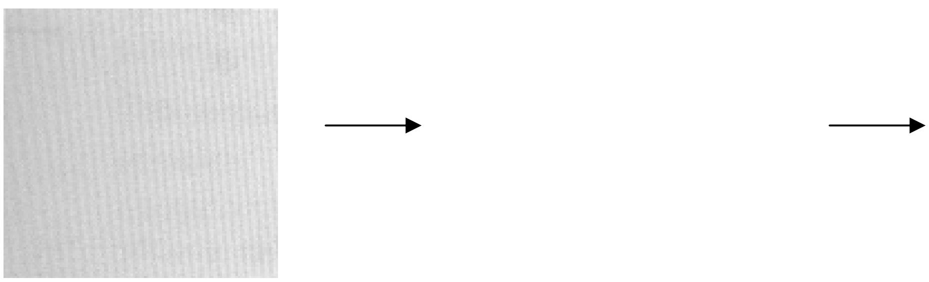
Rule: Divide the square into nine squares whose sides have length $\frac{1}{3}$. Then remove two of the smaller squares, creating the shape of an “H”. Draw the next two figures in this orbit.



Appendix 9

The Sierpinski Carpet
Seed: Square whose side has length 1.

Rule: Divide the square into nine equal-sized squares whose sides have length $\frac{1}{3}$.
Remove the middle squares leaving eight squares. Draw the next two figures in this orbit.

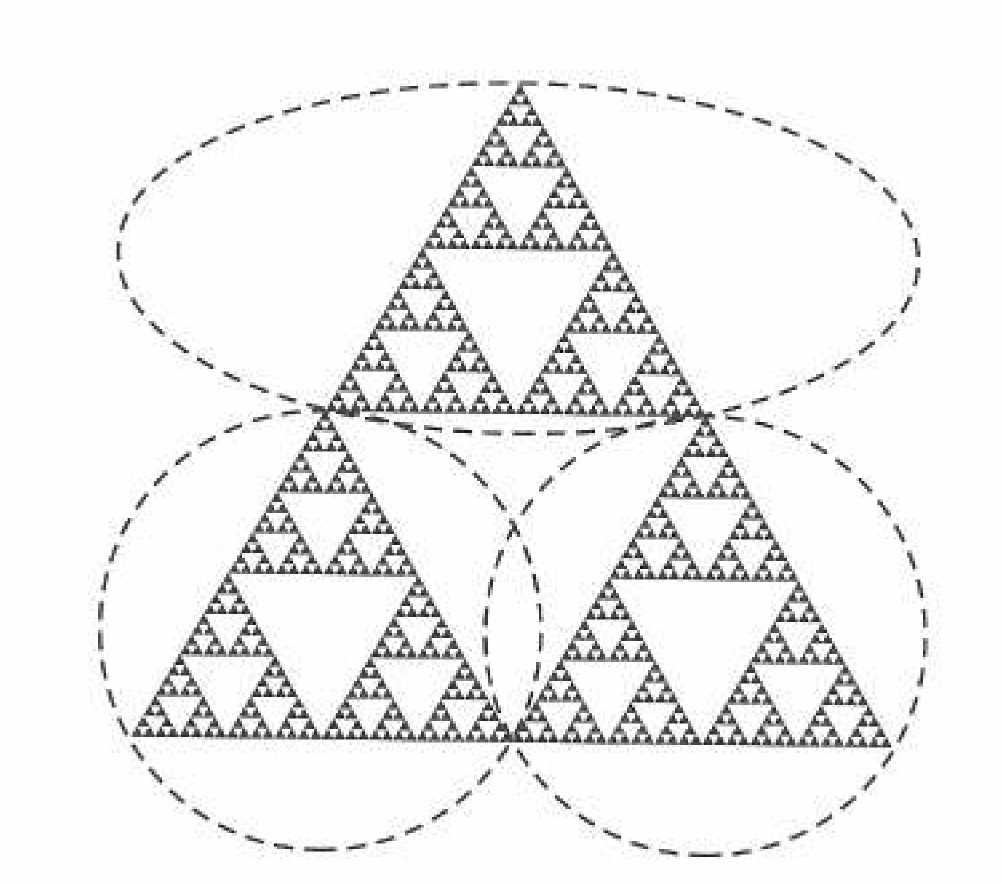


Complete the table below in order to determine the nth iteration.

	Original segment (the seed)	First Iteration	Second Iteration	Third Iteration	nth Iteration
Total number of squares remaining	1	8			
Side length of each square	1	$\frac{1}{3}$			

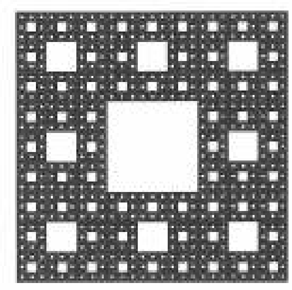
Appendix 10

The Sierpinski triangle



Appendix 11

The Sierpinski Carpet



Examining this figure, find each self-similar piece that has magnification factor 3.

Find a piece of your fractal that demonstrates the fractal's self-similarity at a smaller scale. Record the magnification factor and the total number of pieces at this scale in the last column below.

Magnification factor	1	3	
Number of self-similar pieces	1		

The Fern Fractal

Describe any similarity you see in this figure.

Can you think of other objects in the real world that exhibit self-similarity to some degree? Explain how they are self-similar.

Appendix 12

The Cantor Middle-Thirds Set

Recall the fractal we developed earlier known as the Cantor middle-thirds set.

1. Sketch a picture of this set and circle some of the self-similar regions.

2. What is the corresponding magnification factor that matches the regions you circled?

3. The Cantor middle-thirds set divides neatly into two pieces: one piece in the left-hand interval $0 \le x \le \frac{1}{3}$ and the other piece in the right-hand interval $\frac{2}{3} \le x \le 1$. Label the left endpoint of your Cantor set with the number 0 and the right endpoint with the number 1. Also label the appropriate points as $\frac{1}{3}$ and $\frac{2}{3}$.

4. Look only at the left-hand portion of the Cantor set. List at least eight different numbers that lie in this position.

5. Now multiply each of these numbers by 3 and list them.

6. Do each of these new numbers lie in the Cantor set? Do they all lie in the left-hand side of the Cantor set?

7. Take an endpoint of one of the intervals in the left side of the Cantor at the nth stage of its construction. What happens if you multiply it by 3?

8. Describe what you think of the following statement: Multiplication by 3 is a "magnifying glass" that magnifies the left portion of the Cantor set and yields the entire Cantor set.

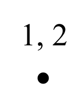
9. Describe a similar magnifying glass that magnifies the left portion of the left side of the Cantor set (the portion in the interval ($0 \le x \le \frac{1}{9}$) and yield the entire set.

10. How about the right portion of the interval $\frac{2}{3} \le x \le 1$? This is tougher. (Try to find an expression that maps 1 to 1 but $\frac{2}{3}$ to 0.)

Appendix 13

The Chaos Game

In the following diagram, play the chaos game with the given vertices. Play with a partner. One of you should roll the die and the other should plot the points and record them. You should choose z_0 somewhere inside the triangle. Then, using your halfway rule and *pencil*, plot the first seven points, $z_0, z_1, z_2, z_3, z_4 z_5$, and z_6, determined by your die rolls. Remember to plot the points very accurately.



• 3, 4 • 5, 6

After plotting z_6, switch to a *pen* and plot $z_7, z_8, z_9, ..., z_{15}$ on the same figure above. Once you have done this, erase $z_0, z_1, z_2, z_4, z_5,$ and z_6.

Record Roll of Die	
$z_0 =$	$z_{21} =$
$z_1 =$	$z_{22} =$
$z_2 =$	$z_{23} =$
$z_3 =$	$z_{24} =$
$z_4 =$	$z_{25} =$
$z_5 =$	$z_{26} =$
$z_6 =$	$z_{27} =$
$z_7 =$	$z_{28} =$
$z_8 =$	$z_{29} =$
$z_9 =$	$z_{30} =$
$z_{10} =$	$z_{31} =$
$z_{11} =$	$z_{32} =$
$z_{12} =$	$z_{33} =$
$z_{13} =$	$z_{34} =$

$z_{14} =$	$z_{35} =$
$z_{15} =$	$z_{36} =$
$z_{16} =$	$z_{37} =$
$z_{17} =$	$z_{38} =$
$z_{18} =$	$z_{39} =$
$z_{19} =$	$z_{40} =$
$z_{20} =$	$z_{41} =$

1. After plotting the first 15 points, describe any pattern at this stage.

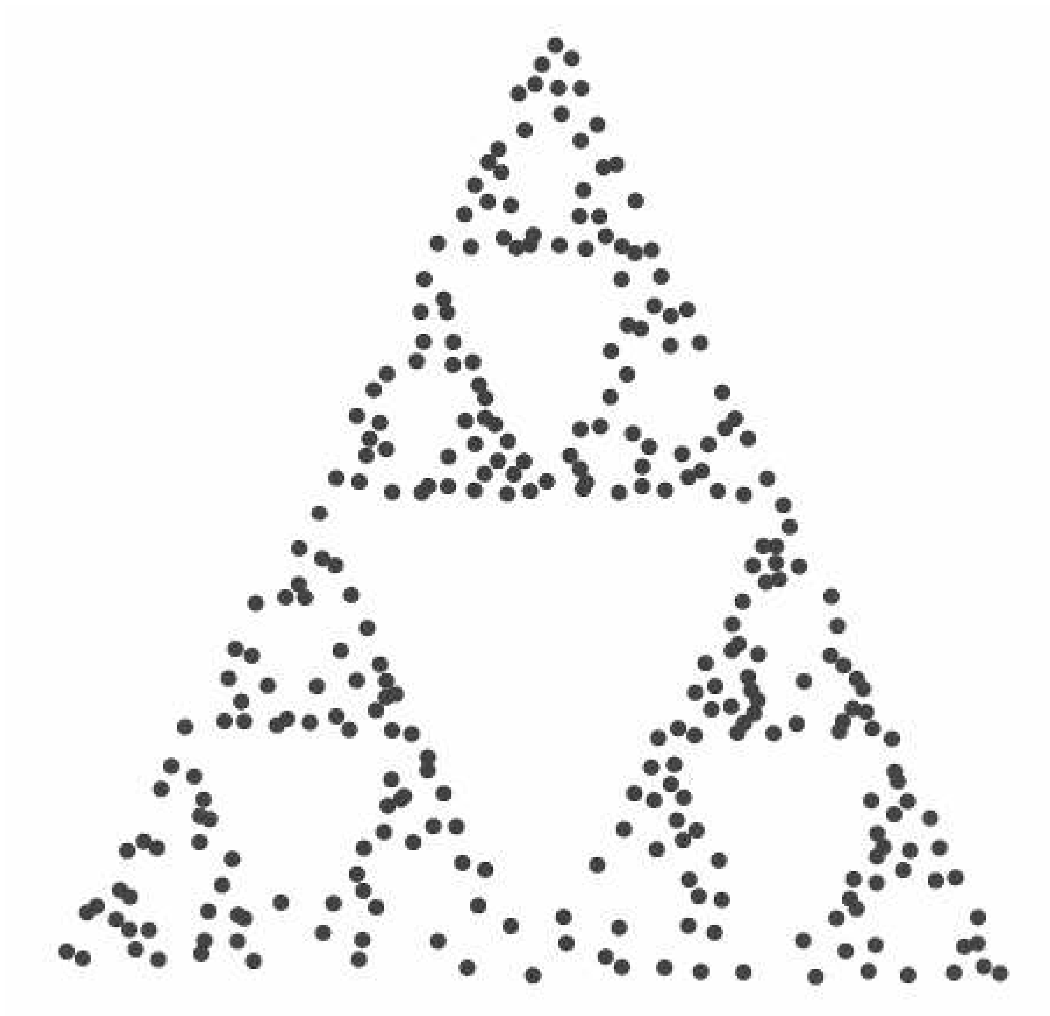
2. Compare your results with those of your classmates. Describe how your patterns compare to theirs.

3. Continue to plot the next 20 or so points in the orbit using a pen. Then, copy the vertices of the original triangle and the points of your orbit onto a transparency while the rest of the class does the same. Do not label the points on the transparency. Bring it up to the overhead projector and put your pile with the rest. Make sure that the vertices of the outside triangle on your transparency lines up with those on the other transparencies.

4. Describe any pattern you see now.

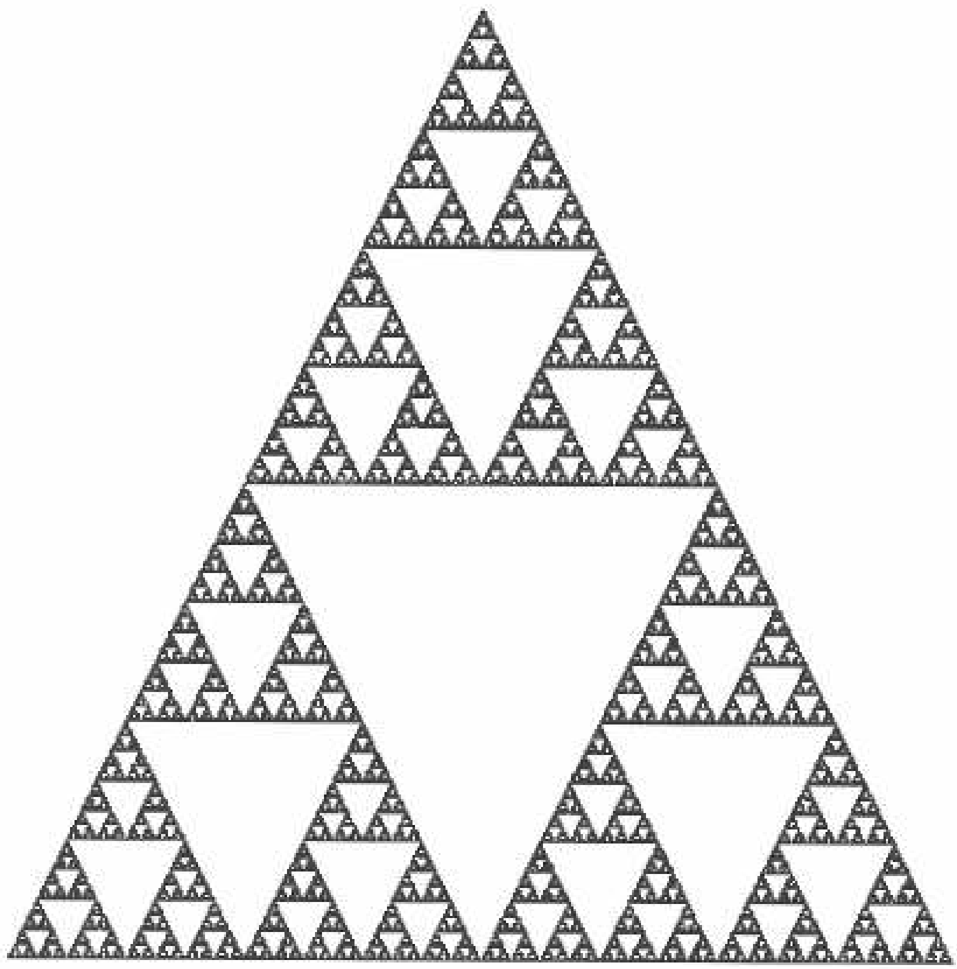
Appendix 14

The Chaos Game – After 300 Iterations



Appendix 15

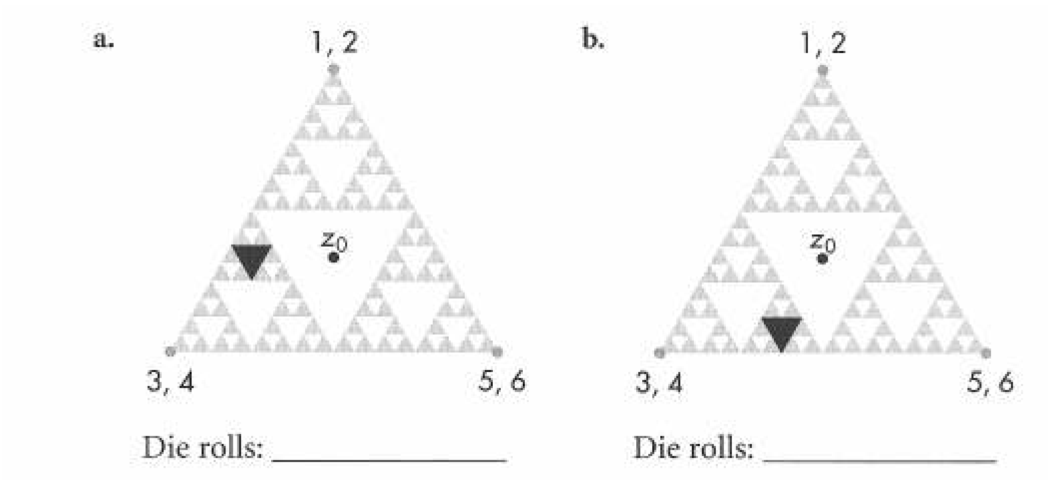
The Chaos Game – After 150,000 iterations (on a computer using smaller dots)



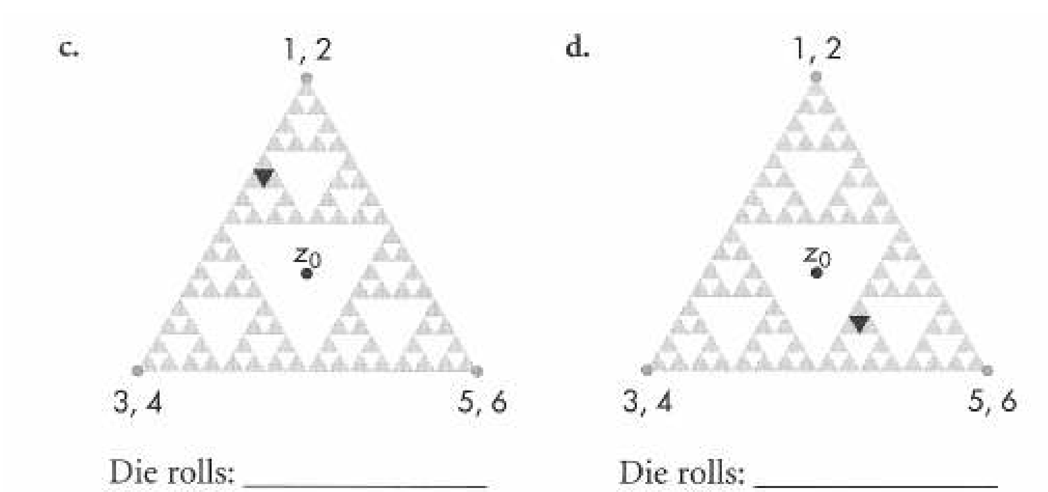
Appendix 16

The Chaos Game – Target Practice

In each of the figures below, suppose you start with z_0 in the center of the middle white triangle. What rolls of the die are necessary in each case for you to land in the shaded target triangles after *two* iterations?



In each of the figures below, suppose you again start with z_0 in the center of the middle white triangle. What rolls of the die in each case will allow you to land in the shaded target triangles after *three* iterations?



Appendix 17

Nonrandom Chaos Game

If the rolls of your die are not truly random, the orbit need not fill up the Sierpinski triangle.

1. Can you predict what would happen if you alternately rolled:
   1, 3, 1, 3, 1, 3, . . . ? Sketch the orbit and explain your findings.

2. How about 1, 6, 1, 6, 1, 6, . . . ? Sketch the orbit and explain your findings.

3. How about 2, 4, 6, 2, 4, 6, 2, 4, 6, . . . ? Sketch the orbit and explain your findings.

After having the opportunity to play the chaos game and examining nonrandom rolls of the die:

4. Why does the Chaos Game which is played with the three vertices materialize into the Sierpinski triangle?

5. Is it necessary to start with a seed that lays inside the triangle?

6. What happens if you start with a seed that lays outside the triangle? Sketch a picture to support your findings.

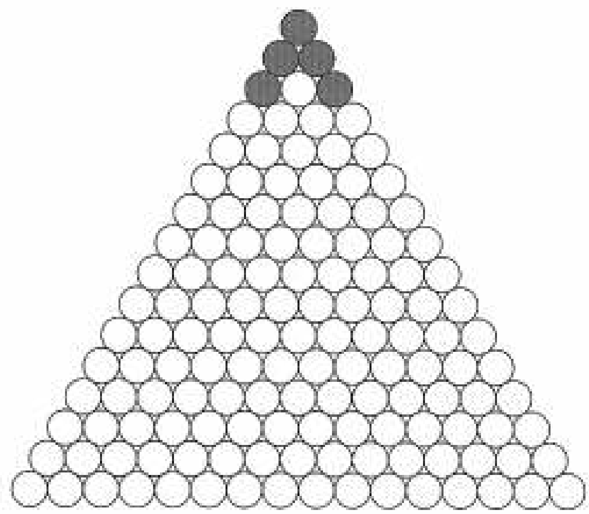
Appendix 18

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
_ _ _ _ _ _ _
_ _ _ _ _ _ _ _

Study the numbers above until you figure out how to generate Pascal's triangle. Then fill in the next two rows.

Now instead of using the whole number in Pascal's triangle, let's change the triangle so that it records only if the entry is even or odd. Start with a triangle filled with circles, as shown below. Shade a circle whenever the corresponding entry in Pascal's triangle is odd, and leave the circle unshaded whenever the entry is even. I have filled in the first three rows to get you started.



1. Do you notice something interesting as you fill in this diagram? Explain.

2. Explain a procedure for deciding whether to shade a circle based on the shading of the circles directly above it.

3. Explain how the procedure you just described is related to the addition of odd and even numbers.

Appendix 19

Famous Fractals

Sketch a rough image of each fractal in the space provided. Then determine for each:

- A number of self-similar pieces that you could divide the fractal into.
- The magnification factor that goes with the number of self-similar pieces you chose.
- The fractal dimension of the fractal.

1. The Sierpinski triangle

2. The Cantor middle-thirds set

3. The Koch curve

Appendix 20

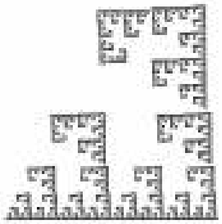
Ordering Fractals by Dimension

Look at the fractals that follow and try to number them according to their relative dimension. Number the fractal you think has the highest dimension "8," and the one you think has the lowest "1." Just to make things tricky, two of the fractals actually have the same dimension.

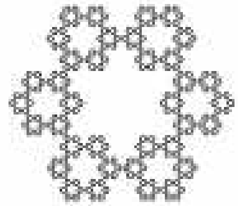
When you are done with your prediction, compute the fractal dimension of each fractal.

Finally, order those fractals according to their computed dimension. How does your ordering now compare with your intuitive ordering earlier?

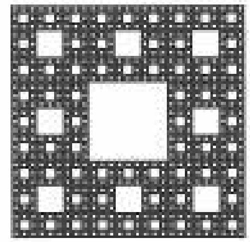
a. ____ dim = ____ ____



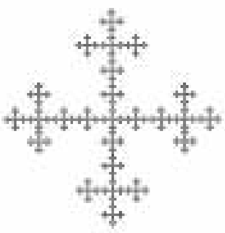
b. ____ dim = ____ ____



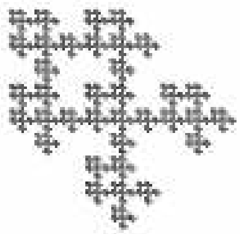
c. ____ dim = ____ ____



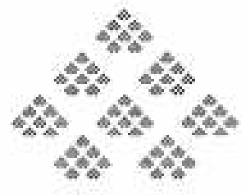
d. ____ dim = ____ ____



e. ____ dim = ____ ____



f. ____ dim = ____ ____



g. ____ dim = ____ ____

h. ____ dim = ____ ____