Making the Connection with the Sierpinski Triangle
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Overview
The Sierpinski Triangle is one of the most well-known and versatile of all fractal figures. The possibilities for rich mathematical explorations and connections are numerous. The middle school Connected Mathematics curriculum is especially suited for this sort of exploration. There is so much for students to revisit and explore: measurement, fractions, area, similarity, and scale factor, to name just a few.

Rationale
One of the most frequent questions that I get from my seventh-graders is "Why do we have to learn this? When will I ever use it?"

Annoying as that is to hear over and over again, it is actually a very astute and reasonable question. It is also a question that I sometimes have a hard time answering. While I can often assure my students that they *will* use fractions and percents and basic mathematical operations many times in their lives, it can be very difficult to explain to a 12-year-old why he needs to learn about scale factor or similarity.

I approached this seminar, *Fractals and Chaos,* with a positive attitude and high expectations. I love learning new things, and I especially love learning about something totally unfamiliar. I had never heard of a fractal. I had heard of chaos, of course; I consider most of my waking hours to be prime examples of chaos. I told my sister-in-law that I was going to take a class on fractals, and she was very excited. "You'll love them! They are so cool!" She showed me pictures. She told me I would learn *so much.* My sister-in-law is working toward her doctorate in plant biology. She is a science person. I got my undergraduate degree in psychology. I am certified to teach high school history, elementary school, and middle school math. I was kicked out of Pitt's nutritionist/dietician program because I failed biochemistry twice. I had to take calculus three times before I finally understood *what* the professor was talking about. I'm not saying I'm dumb, but science and higher level mathematics are not exactly my forte.

So I went to my first *Fractals and Chaos* seminar, and I tried so hard to follow the lecture. We discussed the theory of dynamical systems, Fermat's last theorem, and exponential versus logarithmic rates of growth. I was completely lost. I'd heard of exponential growth, and I vaguely remembered logarithms. What was the use of all this? Why did I need to know about the Canter Set? I felt like I was back at Pitt, suffering through another biochemistry lecture. I felt, I felt…I felt like some of my students must feel when I talk about fractions and decimals and scale factor and similarity!

Maybe I need to feel lost for awhile, I told myself. If I put myself in my students' shoes, if I can relate to what it feels like to be totally confused, I can't help but improve my teaching skills. A little humility is always good.

I kept going to class, week after week, hoping a ray of understanding would strike me. I'm not very good at abstract thinking. My husband loves to watch the Discovery Science channel. He loves all that stuff about black holes and worm holes and String Theory. It makes me nervous. I cannot wrap my brain around it. Give me some good old basic math; *that* I can handle.

Mandelbrot (we'll get to him shortly) has been linked with the fourth dimension and the "space time continuum of Man and Nature." Wow - way over my head.

After a few weeks, though, it all started to get a little clearer. I'm not saying that I ever understood all the complexities and formulas, but I started to read more and more about fractals. There are hundreds of books about fractals and chaos. Type fractals into your search engine, and you could be on the internet for hours. My sister-in-law was right about fractals; they *are* so cool.

In class, we talked about Canter Dust and measuring the coastline of England. I realized that this is an incredibly rich subject. There are so many different directions one could follow while exploring fractals and chaos.

I realized that I could use what I had learned in class for my own classroom. We wouldn't be writing equations and figuring out iterations, but my seventh-graders would love the great pictures of fractals, and I could use one particular fractal, the Sierpinski Triangle, to help make the seventh grade curriculum more accessible and meaningful to my students. Maybe now they would understand why they need to learn about scale factor and similarity!

So what is a fractal? Why are so many mathematicians and scientists exploring fractal geometry and chaos theory?

Fractal geometry and chaos theory give us a new way to look at the world. Instead of looking at objects around us in a linear fashion, fractal geometry gives us a way to describe and analyze the complex objects found in nature. Weather, planetary orbits, human body rhythms, plants, animal behavior, socioeconomic patterns, and even music, can be modeled with fractals.

Computers are, of course, a huge asset when studying fractals. They can solve impossibly intricate formulas and equations in an instant. There are, however, two significant times before the invention of computers when fractals came up as important questions. The first instance was when British map makers discovered the problem with measuring the coastline of Britain. Depending on what unit of measure they used, they got drastically different measurements. It turns out that the closer they looked at the coastline with all its little nooks and crannies, the bigger the measurement became. In fact, there is no way that anyone could ever measure every rock, every pebble, on the rugged coast. Therefore, the coastline of Britain is infinite.

The second instance of pre-computer fractals was when the French mathematician Gaston Julia explored complex polynomial functions. If you're anything like me, your eyes have already glazed over, so I probably don't have to tell you that he specifically looked at polynomials in the form of $z^2 + c$, where c is a complex constant with real and imaginary numbers. The point is that Julia did all sorts of complex formulas on paper, and then years later Benoit Mandelbrot, an employee of IBM, ran some of these formulas on the computer and came up with some really cool pictures.

Benoit Mandelbrot first used the term *fractal* in 1975, almost sixty years after Waclaw Sierpinski created his famous triangle. Fractals are geometric shapes that are self-similar at different scales. They are formed by applying the same procedure over and over again. Some of the most famous fractals are the Koch Snowflake, the Jurassic Park Fractal, and the Sierpinski Triangle.

Mandelbrot is a fascinating character in the world of fractals. He never learned the alphabet, and he never learned his times tables past five. He is now an IBM scientist and a Professor of Mathematics at Yale.

He made his great discoveries by defying the academic mathematics establishment. He went beyond Albert Einstein's theories to prove that the fourth dimension includes the first three dimensions plus all the gaps between them; the fractal dimensions. Fractal geometry is now recognized as the true Geometry of Nature.

The Sierpinski Triangle, sometimes referred to as the *Sierpinski gasket,* is a fractal formed from an equilateral triangle. It was created in 1916 by Waclaw Sierpinski, a Polish mathematician. The beauty of the Sierpinski triangle is found not only in its simplicity, but in its endless complexity and possibilities for exploration.

In a 2005 article in *American Educator,* William H. Schmidt looks at the importance of curriculum in promoting student achievement. Citing evidence from the 1995, 1999, and 2003 Third International Mathematics and Science Study (TIMSS), Schmidt states that "the top achieving countries have coherent, focused, and demanding mathematics curricula."

The handbook for the National Council of Teachers of Mathematics (NCTM) , <u>Principles and Standards for School Mathematics</u>, supports the TIMSS findings and Schmidt's assertions: "A school mathematics curriculum is a strong determinant of what students have an opportunity to learn and what they do learn. In a coherent curriculum, mathematical ideas are linked to and build on one another so that students' understanding and knowledge deepens and their ability to apply mathematics expands." (p.14).

The current middle school curriculum in Pittsburgh Public Schools is Connected Mathematics. The Connected Math Program (CMP) is devoted to developing student knowledge and an understanding of mathematics that is rich in connections. Connected Math calls for an instructional model in the classroom that encourages higher-level thinking and problem solving. With this in mind, I focus on designing lessons that will help the students develop connections with what they have done previously, and use what they already know to explore more complex ideas.

The 2006 Connected Mathematics Project's *Research and Evaluation Summary* states that "All students should be able to reason and communicate proficiently in mathematics. This includes knowledge of and skill in the use of the vocabulary, forms of representation, materials, tools, techniques, and intellectual methods of the discipline of mathematics, including the ability to define and solve problems with reason, insight, inventiveness, and technical proficiency."(p.4).

Working with the Sierpinski Triangle will enable students to develop all these areas of proficiency. The NCTM says that "when challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere."(p.21).

I got the idea for a Sierpinski Triangle curriculum unit from an article I read in the December 2005/January 2006 issue of NCTM's *Mathematics Teaching in the Middle School.* The article was written by Thomasinia Lott Adams and Fatma Aslan-Tutak. They looked at some of the variations in creating Sierpinski Triangles, and explored some of the rich mathematical connections this sort of unit would present to middle school students.

In the sixth grade curriculum, students spend a considerable amount of time exploring fractions, decimals, and percents (*Bits and Pieces I & II*). They also delve into the properties of triangles (*Shapes and Designs*) and revisit area and perimeter (*Covering and Surrounding*). Although I realize that children forget a large percentage of what they have learned, especially over the long summer break, I expect that they will retain some of these basics when they come into my seventh grade math class.

In *Variables and Patterns,* the first unit in the seventh grade curriculum, students describe and generalize patterns, and begin to represent relationships with verbal or symbolic rules. The second unit in the seventh grade curriculum, *Stretching and Shrinking,* introduces the concepts of similarity and scale factor. In *Comparing and Scaling,* we explore ratios and proportions.

All of the aforementioned math skills and concepts can be reintroduced and reinforced through a unit on the Sierpinski Triangle. This curriculum unit involves geometry and measurement, patterns and similarity. Constructing a Sierpinski Triangle will help students in several way; they will have to follow directions carefully; they will have to work slowly and thoroughly; and they will have to develop their higher-level thinking skills. These are all issues that I continuously try to address in my lesson plans.

This unit should take approximately 12 - 15 instruction days. Some of the lessons will take more than one day. The most effective time to introduce this unit would be after the first three seventh grade CMP units (*Variables and Patterns, Stretching and Shrinking, Comparing and Scaling*) have been completed. I realize that the current CMP pacing guide does not leave much room for extras, but this unit could potentially be so rich in connections and reinforcement that the "lost" time would be well worth it.

The true beauty of the Sierpinski Triangle is that it is so very intricate, and yet so simple. Lessons can be designed around it ranging from pre-school to college and beyond.

While researching and preparing this unit, I was reminded of a statement that a fellow middle school teacher once made: "In high school, they teach math. In middle school, we teach kids." Middle school students are a unique breed; they have very short attention spans, they are constantly in motion, and they like to be active participants in their learning. This unit helps them create some real-life applications for math.

Objectives

Before beginning this unit, students should be able to:

- Recognize and draw lines and triangles
- Be able to measure lines and find midpoints
- Understand and use formulas for area and perimeter
- Create, understand, and manipulate fractions
- Use basic computer skills (using a mouse, browsing the web)

By the end of this unit, students should be able to clearly describe a fractal. They should be able to give step-by-step directions for creating a Sierpinski Triangle. After completing all the lessons, students will have reviewed some of the math concepts they learned in sixth grade, such as fractions, area, and perimeter. The seventh grade units they have already studied, *Variables and Patterns, Stretching and Shrinking* and *Comparing and Scaling,* will be reinforced because they will be working with writing formulas, scale factor, similarity, ratios, and proportion.

Math standards 1 through 6 are addressed in this unit plan. In particular, the following math strands are covered:

- **Algebra**: students will understand patterns, relations, and functions *and* students will use mathematical models to represent and understand quantitative relationships.
- **Geometry**: students will apply transformations and use symmetry to analyze mathematical situations.

Ultimately, students will work together to create a Super Sierpinski, which can be hung in the classroom or in the hall for the entire school to admire.

Strategies
The lessons are meant to follow the basic Connected Math strategy of *Launch, Explore, Summarize.* All the activities can be done in cooperative groups of three or four students. Students will be able to help each other remember things they've learned previously, and work to understand new ideas. The *Summarize* portion is especially important because this is when all the math becomes evident.

Classroom Activities
Building Interest
A few weeks before starting this unit, begin to pique your students' curiosity by covering your walls with pictures of fractals. There are literally thousands of beautiful and fascinating pictures to be found; type "fractals" into your web browser and you could spend hours looking at all the incredible examples. The works of graphic artist M.C. Escher are renowned for their intricate repeating patterns. If the students ask about the pictures, tell them that the pictures are all examples of fractals, and they will themselves be creating a fractal in just a few weeks.

Lesson 1: What is a Fractal?

Basically, a fractal is a geometric figure that is self-similar at different scales. It is formed by applying the same procedure over and over again. Show the students specific examples of fractals, such as the Mandelbrot set of fractal images. A demonstration of the Koch snowflake or the Jurassic Park fractal would help students understand the mechanics behind creating a simple fractal.

Mandelbrot set

Creation of a Koch snowflake

Jurassic Park fractal

Lesson 2: Making a Sierpinski Triangle

Materials Needed:

- Triangular grid paper (Appendix B)
- Ruler
- Pencil

After this lesson, students should be able to:

- Define and create a Sierpinski triangle
- Define and apply iterations

The Sierpinski triangle is one of the most simple fractal shapes to create, yet at the same time it can be infinitely complex and interesting.

To create the Sierpinski triangle, the students will first draw an equilateral triangle using the triangular grid paper. This first triangle should be rather large, with its vertices touching the edges of the paper.

After the original triangle has been drawn, the students will begin applying a series of **iterations** (repeating a set of rules over and over). The basic iteration is this:

Connect the midpoints of each side of the equilateral triangle to form four separate triangles, and shade in the triangle in the center (the downward pointing triangle).

At this early stage of the unit, the students will only apply three iterations. The figure that they create will be used to help them answer the problems in the subsequent lessons. By taking the responsibility of making the triangle that will be the basis of the unit, each student will have more of a sense of ownership and connection. Rather than answering questions about a pre-made copy of a triangle, they have been involved in the construction of the triangle from the beginning.

Construction of the Sierpinski Triangle (this shows the 1st, 2nd , and 5th iterations)

Lesson 3: Finding a formula for area

In this lesson, students will:

- Find the areas of the first three iterations, expressed in fraction form
- Formulate a rule to find the area at any given step (step *n*)

Give each student a worksheet with 4 blank equilateral triangles drawn on it (Appendix C). Tell the students to label the triangles 0, 1, 2, and 3. These numbers represent the iterations they applied to their triangle in Lesson #2. They will leave the first triangle blank, and then apply the same process that they did the previous day, except now the iterations have been spread out so that they are easier to analyze.

When they have finished, their triangles should look like this:

0 1

2 3

After the students have filled in their triangles, they will create a chart to record the fraction of each triangle *not* shaded. The completed chart should look something like this:

Iteration	**1**	**2**	**3**	**4**	***n***
Fraction *not* shaded	**3/4**	**9/16**	**27/64**		

The students should now try to figure out what the fraction not shaded would be for the 4th iteration. Some students may get no further than this, and some may not even get this far. Ultimately, some of the more advanced students may be able to come up with a rule for the *nth* iteration.

If the majority of students seem stuck after the 3rd iteration (or even before that), have them look for patterns in the fractions. For the first iteration, the fraction is ¾. The numerator is 3, or 3^1. The denominator is 4, or 4^1. For the second iteration, the numerator is 9, or 3^2, and the denominator is 16, or 4^2. Finally, in the third iteration, the numerator is 27, (3^3), and the denominator is 64 (4^3).

Most of the students at this point should be able to figure out the pattern: for each iteration of the triangle, the fraction *not* shaded is $3^{\#iteration} / 4^{\#iteration}$. So for the fourth iteration, the fraction would be 3^4 (81)/4^4 (256). The general formula for any iteration (n) would be $3^n/4^n$.

If there is time, you could have the students turn the fractions into percents and see if they can find a pattern or a rule.

Lessons 4 - 6: Perimeter, Similarity, Scale Factor and Ratios

To explore perimeter, students could start with an original triangle with sides of 12 units (perimeter = 36) and then break it down into increasingly smaller **similar** triangles and figure out the perimeters. Scale factor could be revisited at this point, and the students may construct a chart something like this:

Scale Factor	**1 (original)**	**1/2**	**1/4**	**1/8**
Perimeter	**36**	**18**	**9**	**4.5**

Once again, the students could look for patterns and make rules and predictions.

Lesson 7: Connecting Sierpinski's Triangle and Pascal's Triangle

Pascal's Triangle could, of course, become a unit project on its own. I decided to add it into my curriculum unit because I found the connection between the two triangles to be fascinating.

The ancient Chinese originally invented it, but it was Blaise Pascal who first discovered many of the patterns in the rows, columns, and diagonals of the triangle. Pascal's triangle is constructed by starting with the number 1 at the tip (Row 0). Row 1 contains 1 and 1, formed by adding the two numbers above them to the left and the right (all the numbers outside the triangle are 0's, so 0 + 1 = 1, and 1 + 0 = 1).

Row 2 is 1, 2, 1 and Row 3 is 1, 3, 3, 1. Each row is formed by adding the two numbers in the previous row.

Provide your students with a blank (or almost-blank; you may want to give them a little help at first) copy of Pascal's Triangle and have them fill it in (see Appendix D). Ask them to look for patterns as they complete the triangle; they may come up with a few new ones!

Here are the first 13 rows:

When all the odd numbers are filled in and the even numbers are left blank, the Sierpinski Triangle is revealed:

Students can play around with Pascal's triangle to make other interesting patterns. For example, if they use divisors other than 2 (for even numbers), such as prime numbers, they will produce different patterns:

Pascal's Triangle divided by 3 to 128 rows

Pascal's Triangle divided by 5 to 128 rows

Pascal's Triangle divided by 7 to 128 rows

The possibilities are endless when working with Pascal's Triangle; the biggest limitation is, of course, time.

Lesson 8: Technology and the Sierpinski Triangle

For a change of pace, you can let the students create a Sierpinski Triangle on the computer. There are all sorts of good sites that recreate Sierpinski Triangles (just go to Google or Ask.com and type in "fractals" or "Sierpinski Triangle").

I found a great Sierpinski fractal **applet** (a software component that runs in the context of another program) at www.arcytech.org/java/fractals/sierpinski. Students can take the triangle to any number of iterations and select a different color for each iteration, creating all sorts of interesting geometric patterns.

Larry Riddle's website (see bibliography) has an IFS (Iterated Function Systems) Construction Kit, where students can construct Sierpinski triangles, carpets, and pentagons, along with other fractals with interesting names like the Heighway Dragon, the Levy Dragon, and the McWorter Pentigree.

Lesson 9: Creating a Super Sierpinski

This is the fun part of the unit, when the students get to show off their creativity and create something that the rest of the school can admire. Each student will create a new Sierpinski Triangle; the number of iterations and the color combinations are entirely up to them.

It may be best to have them all start with the same size triangle, because ultimately you want to piece them all together to create one *Super Seventh Grade Sierpinski Triangle.*

Here is an example of what your students could create:

Source: Larry Riddle, "Sierpinski Gasket"

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Appendix A

Math Standards

1. All students use numbers, number systems, and equivalent forms (including numbers, words, objects and graphics) to represent theoretical and practical situations.

2. All students compute, measure, and estimate to solve theoretical and practical problems, using appropriate tools, including modern technology such as calculators and computers.

3. All students apply the concepts of patterns, functions, and relations to solve theoretical and practical problems.

4. All students formulate and solve problems and communicate the mathematical processes used and the reasons for using them.

5. All students understand and apply basic concepts of algebra, geometry, probability and statistics to solve theoretical and practical problems.

6. All students evaluate, infer, and draw appropriate conclusions from charts, tables, and graphs, showing the relationships between data and real-world situations.

Appendix B: *Triangular Gridpaper*

Appendix C

Worksheet for Lesson 3:
Finding a Formula for Area

Appendix D

Pascal's Triangle