Problem Solving

Lesley M Britton

Pittsburgh Allderdice High School

Contents of Curriculum Unit

Overview

This document contains objectives, guidelines for lectures and discussions, and all problems and prompts needed to guide high school students through an investigation into mathematics critical and logical thinking, and problem solving. The modules in this lesson are designed to reinforce middle school math skills and designed to supplement a first year algebra course carefully aligned with the Pennsylvania mathematics standards.

Objective

For each task in this unit, students will be able to identify and discuss the specific parameters of the problems as well as possible solutions or methods of solving the problem. They will use guided group discussion, small group discussion, and work individually to apply their knowledge of basic mathematics to more challenging problems than the standard curriculum provides. Topics will include ratios and proportions, algorithms for adding fractions and multiplying large numbers, number sense, writing mathematical rules, and using mathematical language.

Rationale

I am planning on using this unit in my Algebra I Mainstream class. These students are, for the most part, ninth graders. However, in most classes there are also several students who are repeating the course in their tenth or eleventh grade year. Regardless of grade level, the students in this course come with a variety of challenges before them. They have varying levels of abilities. Some are very bright and have no problems picking up the material after a day or two, while others struggle to pass. Some students come prepared and ready to learn; others are not motivated to do well in school. This is a dynamic group of students that has the ability to do wonderful things if given the right tools.

One of the major issues that I face as a teacher in the public schools is the constant pressure to keep up with the district-wide curriculum. The Algebra I curriculum is very ambitious and has specific timelines for each topic. It does, however, leave room for teachers to make decisions for their class as they see fit when it comes to assignments and assessments. The curriculum also gives a window (for example 8-10 days) for each unit that sometimes allows time for a small extra task to be included. Unfortunately, this does not happen as often as I would like. It is partly for this reason that I have decided to create my unit as a series of modules rather than a day-to-day unit plan. Modules can be incorporated when a schedule permits without worrying whether there will be time to finish. These modules will be designed so that they can be used on any given day where there might be down time. These modules are designed to provide as much flexibility as possible to the classroom teacher and are designed so that the time span between lessons is not a concern. One lesson is not dependent on another. As long as the basic skills needed for the task have been previously covered, students will have the opportunity to be successful. The flexibility of when the tasks can be used is imperative to me for my Algebra I classes since attendance can be such an issue. Activities built on previous days' work can be problematic with students who were not present for those earlier lessons. With the exception of one (Billiards), these tasks are designed to take no more than one class period. This allows the students who are present an opportunity for enrichment while not putting the students who missed even further behind in the basic curriculum.

The other reason that I write this unit as modules to be used at any time is to provide teachers flexibility in how to use the tasks as teaching tools. My students often show proficiency at a skill on the days when I am teaching it and shortly thereafter. There are many times, however, when I ask them to recall a skill from a month or so ago and they cannot. For example, I spent several weeks this year on the concept of slope as rate of change, y intercepts, and graphing. My curriculum introduced these concepts by having the students work in several different long distance calling plans with varying monthly (y intercept) and per minute (slope) charges. Naturally, they began using phrases like "starting point" and "rate of change" as they discussed which plan to use. Eventually, I introduced slope and y intercept and most were able to relate what they learned through the Calling Plans task to learning the meaning of these words. My students graphed linear equations with ease and I was ecstatic that this new method of teaching these concepts had seemed to work. Finally, we tested on the topic and moved on to the idea of inequality. When I got to graphing inequalities in two variables, I began again with the launch task provided by my curriculum. This task involved finding final scores for figure skaters based on short and long program scores. The students made a graph and placed each skater's score on the graph. Our discussion was centered around who made the team and who did not based on a cut off score of 60. Again, my students really seemed to understand the idea of being above the line or below the line. They even shaded the area where your score needed to lie in order to make the team. Again, I was so pleased. When we moved on to learning to graph two variable inequalities without a situation to link it to, from just being given an equation, my students began to struggle. The first instruction I gave them was to graph the boundary line and to my surprise, the majority of my class could not do it. Even when I simply gave them the equation for the line I wanted them to graph, many simply sat with blank stares. This was quite a learning experience for me, and it brings me to my point that beyond learning the skill at hand, what my students *really* need to learn is the basic skill of problem solving. My students need to learn to think mathematically. They need to get out of their heads that there is a step by step process of solving each kind of problem and begin to look at each problem as a puzzle that needs to be solved. Everyone isn't going to solve it the same way. Everyone isn't even going to think about it in the same way. What I really want to teach my Algebra I students is to apply and build on the skills that they have in order to make that skill set larger over the years. When teaching, I often refer to their "mathematical tool-belt" as the set of skills they have and things they know that they can use to solve problems. Our goal is to continually add tools to this belt, but not forget about the old tools that are already there.

Along with the content standards that every curriculum focuses on, The National Council of Teachers of Mathematics focuses on five process standards for children in pre-kindergarten through high school graduation. These standards have goals for all levels and it is important for teachers to focus on these standards as their students grow mathematically. Students need to be able, for example, to use the language of mathematics at all levels. They need to know what addition is, be able model it, use the words that describe the operation, and apply the operation correctly to real world problems. For me, this is a key goal for all students, regardless of age or ability level.

As I go through the school year with my ninth graders, I have the opportunity to watch them grow in many ways. Sometimes, I get the pleasure of seeing them achieve what is often referred to as "algebra readiness". According to Pearson, the world's leading education publisher, "Students become "algebra ready" by gaining a deeper understanding of multiplication, fractions, problem solving and critical thinking" (**"**Algebra Readiness"). Until they achieve this, my instruction and classroom activities can have very little meaning to them. The idea that these problem solving tasks can be used at any time throughout the school year benefits those students who might achieve algebra readiness later than others did. Teachers can choose to purposefully wait to use a task until later in the year. This provides students who learned it an opportunity to pull it back out of their mathematical tool-belt and use it again, reinforcing the concept in their mind. For others who did not master the topic the first time, it gives them another chance for learning by spiraling the topics to a later place in the curriculum. For example, if slope didn't make sense to a student back in October when the class first learned it, working on a task where students are required to interpret slope can provide them another opportunity to understand slope, attach meaning to it, and therefore, place that concept once and for all into their mathematical tool-belt.

Presenting students with challenging yet solvable problem solving tasks can also help students to learn by helping their teachers to know them (and their learning style) better. The first time I ever gave my Algebra 1 students a challenging problem solving task, I was amazed at how much I learned about them in a very short time. I quickly learned about their frustration levels. Some students surprised me quite a bit. Students who can struggle to stay awake day to day were suddenly racing to the board to tell the class what they could contribute to the solution. These types of problems interest them *because* they are so challenging. This helped me identify the students who are shutting down and failing because they are bored, not because they lack ability. I found out that some of my students have amazing number sense. I discovered that some of my students thrive on feeling like they've achieved something beyond what is minimally expected of them. I learned that others did not like being challenged and that some struggle without specific instructions or guidelines. As I give my students more and more different types of problem solving tasks, I watch each of them rise to the occasion when it is a task that their specific skill set allows them to succeed with. These tasks have not only helped them to develop the types of skills and thought processes that I, as a teacher, want them to have, but they have given them the confidence of knowing that they are smart and capable of solving even the most challenging problems.

The problem solving tasks in this unit are designed with such flexibility that they can be used to immediately strengthen and deepen understanding of a topic when it is being taught or used later to reinforce, re-teach, or challenge students. In whatever capacity they are used, they are specifically geared toward teaching students not only to follow instructions and read problems carefully, but to use all of the tools they have at their disposal. I want my students to use what they know, have an awareness of what they don't know, ask questions when they need to, utilize each other's knowledge, and challenge themselves so that they can learn to be thinkers. These skills, when mastered, will serve them far beyond any mathematics classroom and into a challenging outside world.

Dr. Randall Charles[1] links mastery of algebra and math skills to the long-term fiscal health of our economy and future generations. He contends that "Virtually all of the research in recent years confirms that in order for our students to compete in the global 21st century economy, they simply must improve significantly in their understanding of math concepts and skills. Mathematics is a significant obstacle for many students, and this is no longer acceptable"("Algebra Readiness"). My students need to learn to be problem solvers. It is not enough for them to memorize algorithms and simply get by. By teaching problem solving, teachers can create logical thinkers that learn not to give up when something seems difficult or insurmountable. We can teach our students how to use their knowledge and work with others to find solutions to problems of any kind.

Task 1: Billiards

The Billiards task (see Appendix A) requires basic skills in writing and reducing ratios, drawing conclusions, writing and applying rules, and using mathematical language. This task can be used as a follow up for teaching ratios and proportions, a re-teaching exercise, or later in the year as an activity to spiral the idea of ratios and proportions back into the curriculum.

In this task, students will use a set of ground rules to create the set of data that compares the dimensions of a pool table to the number of time a ball bounces before it lands in a corner pocket. Students use graph paper and a ruler to draw each pool table and sketch out the path of the ball. Drawing each of the situations out allows students to "see" the situation a little more clearly which will help them to eventually draw conclusions and find patterns. The Billiards task

[1] Dr. Randall Charles is a Professor Emeritus in the Department of Mathematics at San Jose State University, past vice president of the National Council of Supervisors of Mathematics, and author of four of the Pearson math programs approved for adoption by California school districts

has students write out ratios that represent the dimensions of the pool table, reduce these ratios, and eventually use them to write conjectures about how the dimensions of a pool table can be used to predict the path and ultimate landing place of the ball. Students will work as a class to come up with a set of rules for making predictions, test those rules for validity, and ultimately use them to make predictions on pool tables that are much larger than might be drawn on standard graph paper.

In addition to algebra standards, the Billiards task quickly addresses the Representation content standard by having the students draw out each situation. It goes on to address the Reasoning and Proof content standard as it encourages students to make conjectures and then test them for validity. During class discussion, students will compare and contrast the different ways they found to prove their conjectures. The Communication standard is addressed as students discuss their arguments and conjectures using mathematical terminology. It is here where the teacher has an opportunity to encourage accountable talk and the use of appropriate terms like ratio, proportion, addition, conjecture, and many others.

Task 2: Number Plays

The Number Plays task (see Appendix B) requires only that students be able to add positive integers. Since the skill level required for this task is low, it is a task that can be used at any time during the year (or even on the first day of school). Since students only need to know how to add to complete the task, it may seem easy at first. However, as students work their way into the task, it requires them to use skills like problem solving and reasoning to figure out just how they can arrange the numbers to fit the given criteria. Since there are many solutions to both tasks in this set, this is a task that can be used over and over throughout the year.

In the first part of the Number Plays task, students are asked to fill in nine circles with the integers one through nine in order to produce a true addition problem. In the second part of the task, students use the numbers one through ten to fill in a pyramid of circles. In this case, the number placed in each circle must be equal to the difference of the numbers in the two circles that it rests on.

This task addresses content standards of Reasoning and Proof (as the students are asked to figure out patterns and rules for how numbers can and cannot be placed), Problem Solving, and Communication. Since there are many different solutions, this task provides a good opportunity to work in groups, share solutions, and allow students to have the opportunity for success even if they’re not the first to find a solution.

Task 3: A’s and B’s Are Numbers Too

The A’s and B’s Are Numbers Too task (see Appendix C) requires basic skills in addition, subtraction, multiplication, and division. The first part of the task requires students to have knowledge of the algorithms used to add fractions as well as the idea of lowest common denominators, and reducing fractional answers to lowest form. The second part of the task requires students to apply their knowledge of the algorithms used to multiply large numbers. This algorithm is referenced in two different ways; once with the entire product given but the multipliers are not, and once with partial numbers provided in both the multipliers and the product. These problems provide students the opportunity to understand the algorithm on a higher level than is required to simply use the algorithm for basic multiplication.

Since this task both challenges students with a non-traditional way of looking at multiplication and asks them to connect their knowledge of multiplying numbers and products to the algorithm that is commonly used, it nicely addresses both the content standards of Problem Solving and Connections.

Strategies and Classroom Activities

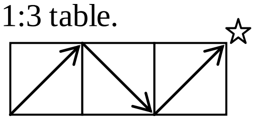
This section provides instructions and ideas for implementing these tasks into your curriculum. You will find ideas for introducing tasks, common student misconceptions to look out for, and possible solutions and examples for each problem solving task.

Task 1: Billiards

For this task, each student will need several sheets of graph paper and a ruler. A nice introduction to this task is a clip from the Disney video "Donald in Mathmagicland" where Donald Duck is learning about the mathematics involved in playing a particular billiards game. This portion of the video can be found online[2] and is about five and a half minutes in length. In this clip, the narrator explains how billiard players use simple mathematics to predict the path of the ball they are about to hit. Though this task does not have students do these same calculations, it is a nice way to introduce the idea of mathematics in a game that students are familiar with.

After viewing the clip, read the instructions for the task as a class. It is important that students understand not only what they are doing in order to collect data but also what information they are to be collecting as they go. It is also a good idea to do the first few together. (Doing just the first rectangle together is not helpful since it is only a one by one and the ball simply goes across the table and into the upper right corner pocket. I suggest doing the one by one in addition to another rectangle that has non-square dimensions.) An example and some suggestions for sketching paths consistently are below.

Example sketch for a 1:3 table.



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[2] http://youtube.com/watch?v=qLTKX4WxmZ4&feature=related

- ✓ Have students label each diagram on their graph paper with the dimensions of the table as they are listed in the data table (1:1, 1:2, etc.). This will enable them to easily reference their diagrams as they begin to investigate patterns and draw conclusions.

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- ✓ Encourage students to take their time and be as neat as possible. Straight lines and 45 degree angles are the key to good data for this task.
- ✓ <u>Always</u> start in the lower left corner. If students change their starting point they will not have the same conclusions as the rest of the class. (This might be an experiment for another day of investigation. How do your conclusions change if you start in another corner?)
- ✓ To ensure students are drawing the paths of the balls at 45 degree angles, have them check to make sure their path goes directly through the line intersections on the graph paper.
- ✓ Have students put arrows on the lines showing the direction of the ball as it moves across the table. This makes it easier to re-trace a path if needed.
- ✓ Remind students to stop as soon as the ball lands in a corner and place a star in that location.
- ✓ The first "hit" for each ball is the lower left corner (the first hit is what makes it start moving). Subsequent "hits" are when the ball bounces off of a table side OR lands in a pocket. For example, in a one by one table, the ball would be hit in the lower left corner, travel diagonally across the table, and land in the upper right corner. We would say that this ball had a total of 2 hits. In a one by two table, the ball would be hit in the lower left corner, bounce off of the right side of the table, and then land in the upper left corner. We would say that this ball had a total of 3 hits.
- ✓ As soon as students are done sketching the path for a particular table have them record that data (number of hits and pocket it landed in).

After carefully sketching the path and recording the hits and landing place for each billiard table, students should have a set of data large enough to draw some conclusions. If they haven't done so already, be sure that students have written each length to width ratio as a reduced fraction in their table. The reduced ratios are imperative in order for students to be able to easily draw conclusions from their data.

Examples: 2:2 will be re-written as $^{1}/_{1}$

2:4 will be re-written as $^{1}/_{2}$

The questions under "Conclusions" are designed to guide students to look at the correct pairings of data in order to draw conclusions. If you have an advanced group of students you may prefer not to use these guiding questions. For a more advanced group of students, you may want to give them the rules and the scenario and see what they can come up with. You can use these as guiding questions within groups if necessary.

For your reference, the completed table is below.

Length to Width Ratio	**Write the ratio as a REDUCED fraction…**	**Number of hits**	**Pocket it landed in (upper right, lower right, upper left, or lower left)**	**Length to Width Ratio**	**Write the ratio as a REDUCED fraction…**	**Number of hits**	**Pocket it landed in (upper right, lower right, upper left, or lower left)**
1:1	$\frac{1}{1}$	2	UR	3:1	$\frac{3}{1}$	4	UR
1:2	$\frac{1}{2}$	3	LR	3:2	$\frac{3}{2}$	5	LR
1:3	$\frac{1}{3}$	4	UR	3:3	$\frac{3}{3} = \frac{1}{1}$	2	UR
1:4	$\frac{1}{4}$	5	LR	3:4	$\frac{3}{4}$	7	LR
1:5	$\frac{1}{5}$	6	UR	3:5	$\frac{3}{5}$	8	UR
1:6	$\frac{1}{6}$	7	LR	3:6	$\frac{3}{6} = \frac{1}{2}$	3	LR
1:7	$\frac{1}{7}$	8	UR	3:7	$\frac{3}{7}$	10	UR
1:8	$\frac{1}{8}$	9	LR	3:8	$\frac{3}{8}$	11	LR
2:1	$\frac{2}{1}$	3	UL	4:1	$\frac{4}{1}$	5	UL
2:2	$\frac{2}{2} = \frac{1}{1}$	2	UR	4:2	$\frac{4}{2} = \frac{2}{1}$	3	UL
2:3	$\frac{2}{3}$	5	UL	4:3	$\frac{4}{3}$	7	UL
2:4	$\frac{2}{4} = \frac{1}{2}$	3	LR	4:4	$\frac{4}{4} = \frac{1}{1}$	2	UR
2:5	$\frac{2}{5}$	7	UL	4:5	$\frac{4}{5}$	9	UL
2:6	$\frac{2}{6} = \frac{1}{3}$	4	UR	4:6	$\frac{4}{6} = \frac{2}{3}$	5	UL
2:7	$\frac{2}{7}$	9	UL	4:7	$\frac{4}{7}$	11	UL
2:8	$\frac{2}{8} = \frac{1}{4}$	5	LR	4:8	$\frac{4}{8} = \frac{1}{2}$	3	LR

By comparing their reduced ratios to the number of hits it took before the ball landed in a pocket, students should find that adding the numerator and the denominator of the reduced fraction gives them the exact number of hits.

Examples: 2:2 reduces to $\frac{1}{1} \rightarrow 1+1=2$ hits

4:6 reduces to $\frac{2}{3} \rightarrow 2+3=5$ hits

By comparing their reduced ratios to the corner the ball ultimately lands in, students should find that the even and/or oddness of the numerator and denominator determine this. Each different ratio will fall into one of three categories: $\frac{odd}{odd}$ lands in the upper right corner, $\frac{odd}{even}$ lands in the lower right corner, $\frac{even}{odd}$ lands in the upper left corner. ($\frac{even}{even}$ will always reduce to one of the other 3 circumstances.)

Through guided group discussion, students should notice these patterns on their own. Have someone record all of the conclusions from the class discussion on the board (or large postable paper). Once everyone has shared their conclusions, have the class come up with a cohesive set of rules for determining the number of hits and the landing corner for any table based only on its dimensions.

Once the class has agreed on a set of rules, have them complete the "Making Predictions" section of the task. In this section, they are given table dimensions and asked to use their set of rules to predict what the outcomes for hits and landing corner will be. After making their predictions, they draw out the path of the ball to verify. In the latter questions, they are given table dimensions that are larger than one might like to sketch. For these, it will be handy for them to use their new set of rules exclusively.

If time permits, this is a situation where you can easily allow students to make up a problem of their own. Have students take turns making up dimensions for the rest of the class do predictions for.

You will need at least 2 days of class to properly complete and discuss this task. If you have students who will work outside of class, it is reasonable to give them a day to begin work on drawing their tables and filling in their data table. During this class period, make sure that each student is following the algorithm correctly. Otherwise their data will not be correct and will lead to inaccurate conclusions and predictions. Once you are sure students are drawing the pool tables and paths of the balls correctly, you may assign them to finish drawing the tables and come up with some conclusions for homework. Regardless of the time frame you allow your students for completing the task, be sure to allow a significant amount of time (a second class period) for discussion and for the class to agree on what they believe the "rules" are for determining the hits and landing corner. The "Making Predictions" portion of the task may also be assigned for homework.

This is a task that most of my students really enjoy doing. They like having the success of being the first to see a pattern, finding that they remember proportions and ratios and playing teacher for a moment, being good at making the drawings, or finding that they are good at communicating what others in their group are trying to say. Because this is so, I try to take as much time as I can to get my students talking about this task. There are many different ways that someone can contribute to this problem situation over a two day period and I find that many of my students rise to the occasion when given the opportunity. If time permits, I highly recommend taking the time to allow your students to do this task in groups and discuss as a class.

Task 2: Number Plays

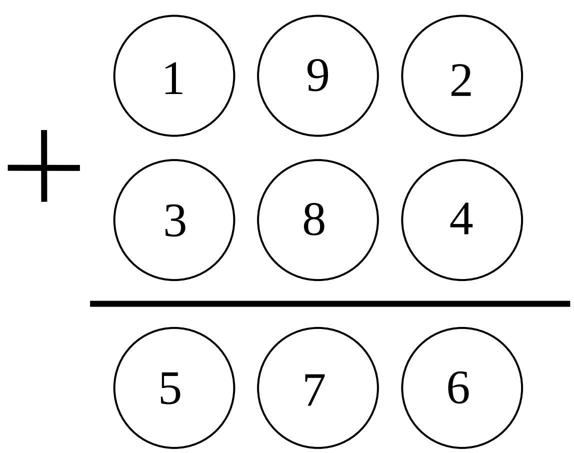
The Number Plays task has two different, yet similar, “games” in it. Since there are several solutions to both of them, they allow many students to experience success instead of just the first student who finds a solution. I give these tasks to my students early in the year and find that I can learn a lot about their basic skills and number sense by how they handle this task. Since these are such important skills for learning Algebra, I find this a very useful task for early in the year. For the convenience of the teacher, sheets with multiple “boards” for each exercise are included with the task. These are easy to reproduce and students may need several of them as they go through the solution process.

The first task asks students to arrange the numbers one through nine in an addition problem where two three-digit numbers are being added and the sum is a third three-digit number. The trick to this problem is that each number may only be used once. Most students will think it looks easy at first, but it certainly is not. What is very nice about this task is that it is very challenging (even for a roomful of math teachers) but doesn’t require a lot of mathematical knowledge. When I give this task to my students I am able to work alongside them to try to find solutions.

Allow students some time with this task. They will naturally begin using only trial and error and should be able to find a few solutions this way. Encourage discussion among the class and try to get students to share ideas about what works and doesn’t work. For example, since you can’t carry a one when you’re in the leftmost column, the nine cannot be placed in that row (because anything added to nine would be a two-digit number and would require the carrying of a one).

As students begin coming up with solutions, classify their solutions by writing them horizontally with the smallest number first. This will avoid students repeating solutions and also allow them to begin to see any patterns in what the solutions look like.

Example: Rewrite the solution below as $192 + 384 = 576$

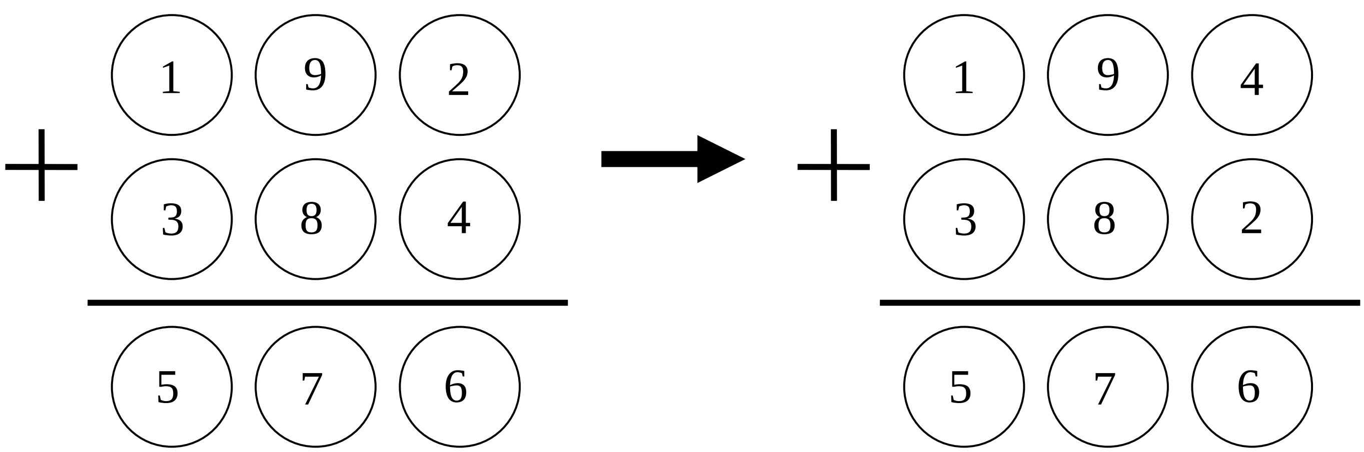


If solutions are classified with the smallest addend first, there are 168 original solutions to this problem. Writing each solution on an index card is a good idea because it not only helps you stay organized, but also allows for easy storage of the solutions. Since there are so many solutions, this is a task that can be pulled back out many times throughout the year and students can continually find more and more solutions.

When you have several solutions, have students arrange them (horizontally written with the smallest number first) in numerical order (smallest to largest or vice versa). Center a class discussion about any patterns that exist in the solutions. As they proceed, students may notice that the nine can never be in the leftmost column (since adding a 9 to any number would produce a 2 digit number). Make a point of this as it is very helpful in finding other solutions.

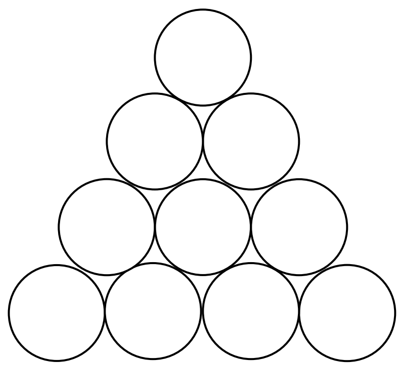
Once students find one solution, center some discussion around how you might find other solutions from this one. By simply exchanging the numbers in the addends, an entire family of solutions may be found. For example: $192 + 384 = 576$, the original solution, can be used to generate others by switching the 2 and the 4, the 9 and the 8, the 1 and the 3, or any combination of these.

> Example: By switching 2 numbers in the same column, the original solution becomes a new solution in the same family.

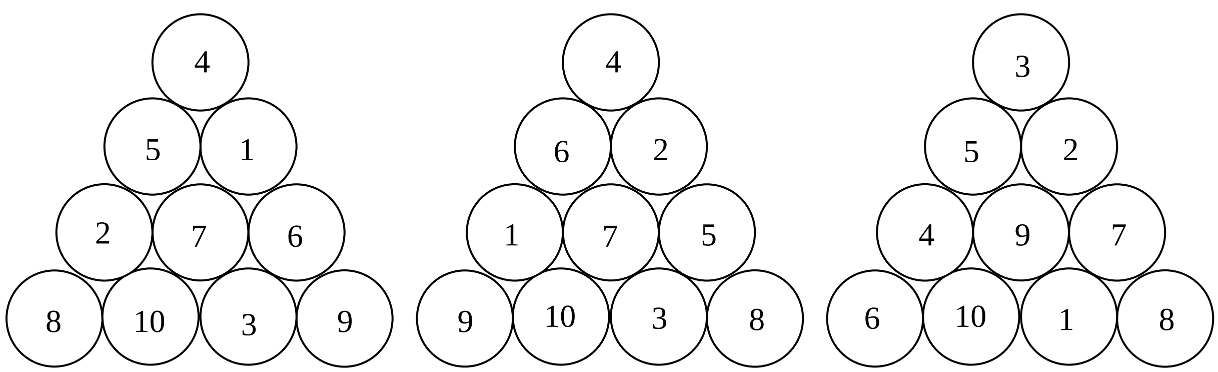


If time permits, a short discussion could also be centered around the digital root[3] of the answers. For the final answers in these problems, the digital root is always 9. In the example above, 576 is the final sum and has a digital root of 9 ($5+7+6=18$ and $1+8=9$).

The second task is similar to the first in that it asks students to arrange numbers in a set of circles. This task, however, has very different rules. In this task, there are ten circles arranged in a pyramid as seen below. Students are asked to use the numbers one through ten to fill these circles in (using each number only once). The only rule is that the number placed in each circle must be the exact difference of the numbers in the two circles that it rests on. For example, if a five is placed in the top circle, the two circles below it could contain a one and six, a two and seven, a three and eight, or a four and nine. It could NOT contain a five and ten because the five has already been used.



Some examples of possible solutions include:



Have students post their solutions as they find them. I find that often, students may think they have a solution, but in reality they have used a number twice or made an error in subtraction at some point. To avoid keeping non-solutions, have students place their solution on the board and have the rest of the class check it to see if they agree. It's a nice way to get students involved

[3] The digital root of a number is found by finding the sum of the individual digits of the number, finding the sum of the individual digits of that number, and so on, until a single-digit number results. For example, the digital root for 246 would be 3 since $2+4+6=12$ and $1+2=3$.

who may not be able to find a solution themselves, but are able to identify a solution when they see it. As you begin to get solutions, encourage class discussion of what the solutions look like. Be sure that students notice that the ten must always be in the bottom row and clarify the reason for this. Ten is not the difference of any of the one-digit numbers and must, therefore be used as a number to be subtracted from and not a difference itself.

Task 3: A's and B's Are Numbers Too

This task has three individual problems in it. They may be used individually or as a group.

The first problem gives students a fraction addition problem. Before presenting this problem (regardless of the level of the students) I find that it is helpful to review the process of adding fractions. Students even in the highest levels of mathematics tend to let simple rules like finding a common denominator slip their minds at times so review can be helpful before beginning this task.

The problem is simply stated as $\frac{1}{3}=\frac{1}{A}+\frac{1}{B}$ and students are asked to find the values for A and B. Different students will approach this problem in different ways. Encourage discussion and allow students to explain how they would go about getting an answer. There is certainly more than one way to find the answer to this problem. If your students are stumped, jumpstart the discussion by talking about how the numerator of 1 is possible. In order to add two fractions you must first have a common denominator (that's going to be the tricky part). After that you simply add the numerators. Ask the students what this must tell them about the answer of $1/3$. With some discussion (and maybe a little prompting since fractions can often be a sticking point), students should realize that this fraction has to be a reduced form of the original answer. From there, a discussion of possible denominators should follow. Guiding questions for this section might include asking what fractions we know that reduce to $1/3$, if the original denominators were bigger, smaller, equal, or multiples of 3, and where the unreduced denominator came from.

A nice way to investigate is by guessing only one value (for either A or B) and then using subtraction to find out if a value for the other variable works out. For example, you could guess a value of 5 for A. If this is the case then you would have…

$$\frac{1}{3}=\frac{1}{5}+\frac{1}{B}$$

so...

$$\frac{1}{3}-\frac{1}{5}=\frac{1}{B}$$

An easy calculation tells us that $\frac{1}{3}-\frac{1}{5}=\frac{5-3}{15}=\frac{2}{15}$. Discuss with students why, based on this calculation, 5 could not possibly be a value for A or B. (The numerator in the reduced fraction is 1. Our fraction has a numerator of 2 and is not reducible.)

Eventually, students should discover that a value of 4 for either A or B will produce a correct answer. When plugged in for either A or B a value of 12 results from the equation.

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{B}$$
$$\frac{1}{3}-\frac{1}{4}=\frac{4-3}{12}=\frac{1}{12}$$

After students have used discussion, guess and check, and other problem solving methods to determine the values for A and B, you may point out the mathematical rule that applies to all unit fractions: $\frac{1}{N}=\frac{1}{N+1}+\frac{1}{N(N+1)}$. Show that this equation works for our situation and then give them similar problems to try the rule out on.

Example: $\frac{1}{5}=\frac{1}{A}+\frac{1}{B}$

Since $N=5$ in this case, A must be $N+1=6$ and B must be $N(N+1)=5(6)=30$

Check it and see that $\frac{1}{6}+\frac{1}{30}=\frac{5+1}{30}=\frac{6}{30}=\frac{1}{5}$

The second problem in this task is another A and B problem that address the algorithm used to multiply multi-digit numbers.

Students are given the problem below and asked to find the values of A and B.

		A	**B**
	x	**B**	**A**
	1	1	4
3	0	4	
3	1	5	4

Before presenting this problem, it is a good idea to give the students a straightforward two-digit by two-digit multiplication problem and discuss with them the process by which they solved it. Point out that we really split this type of problem into two smaller problems. We take one number and multiply it by the number in the one's place and then by the number in the tens places of the other number. We then add these two products together. In the example below you can see that we first find the product of 32 and 5. This product of 160 is written in the first line. Then, we are taught to put a zero "place holder" in before multiplying by the 1. Point out to your students the reason for this. We're not really multiplying by 1; we're multiplying by 10. The product of 32 and 10 is 320. This number is written in the second line and we can then easily add the two products to find the final answer of 480.

$$\begin{array}{rrrr} & & \mathbf{3} & \mathbf{2} \\ & \text{x} & \mathbf{1} & \mathbf{5} \\ \hline & 1 & 6 & 0 \\ & 3 & 2 & 0 \\ \hline & 4 & 8 & 0 \end{array}$$

This multiplication problem can also be written out horizontally to relate it to the distributive property.

$$32(5 + 10) = 160 + 320 = 480$$

Once students understand the process of multiplication well, introduce the AB problem to them and ask them to apply those same concepts to this problem. Have students discuss and during this discussion be sure to point out the following:

- ✓ The product of AB and A must be 114
- ✓ The product of AB and B must be 304
- ✓ A must be less than B (since its product with AB is less than B's product with AB)

Based on these 2 ideas, point out that both A*B and B*B must be products ending in 4. Have your students list possible single digit values for A and B (1 and 4, 2 and 2, 3 and 8, 4 and 6). From here, students may guess and check to find which of these combinations fits the problem.

As an extension once the class understands the solution, have students make up their own AB problem. They choose the numbers, work out the product, and write their problem in the same format as the problem above (Keeping their A and B values to themselves). The rest of the class can then try to figure out their values for A and B. This is an easy problem for them to create and their classmates, based on the one they've just completed as a class, should be able to achieve success in finding the solution to one of their classmates' problems. I find that my students really enjoy having the opportunity to make up a problem of their own and having this opportunity allows them to be more involved in their learning.

The final problem in this task is another multiplication problem that addresses the same algorithm as problem number 2. In this problem, however, students are given part of each number, part of the partial products, and none of the final answer. It is their task to figure out what the original numbers and their product is.

```
      4 _ _
  x     _ 7
  ---------
    _ _ 8 2
  1 2 _ _
  ---------
  _ _ _ _ _
```

As in the last problem, preface this problem with a standard multiplication problem so that you can review multiplication algorithms and the reasoning behind them. Once students are presented with the problem situation, note that we can begin to piece the problem together using what we know about the multiples of 7. To get the first partial product, we must multiply 7 by the mystery number on top. When we multiply 7 by the number in the one's place of this mystery number, we get some number that ends in 2. Since this is fairly straight-forward, students should be able to look at the multiples of 7 and figure out that the only one digit number that could go in that place would be a 6.

With a 6 in the one's position of the first number, we know that we will carry a 4 into the ten's position. Now we will have $(7 * ?) + 4$ equal some number that ends in 8. Again, students can look at the multiples of 7 (knowing now that they need one that ends in 4) to determine that the only number that would fit this position is a 2.

Once a 2 is placed in the ten's position of the first number, the first partial product can easily be filled in. $(2 * 7) + 4 = 18$, carry the one to the next column and you have a 29 to fill in the first two blanks of the product.

To figure out the second partial product note that you are taking 426 multiplied by some number and getting something slightly larger than 1200. From this, students should be able to estimate what number goes in that blank. (The number cannot be any larger than 3 or the product would be much larger than 1200.) Using guess and check, students can figure out that the missing number in the second must be a 3. Once both partial products are complete, the problem is solved using simple addition of the partial products.

This problem, like the A and B multiplication problem addresses the algorithm of multiplication without introducing variables into the problem. If my students are having trouble with the A and B problems, this is a problem that can be used to step them into it. Since they are given numbers in more than one place it allows them more opportunity to see a starting point without being overwhelmed by what appears to be a very difficult problem.

Appendix A

Billiards

For this activity, you are going to draw several pool tables that have different length to width ratios. (You'll need some graph paper and a ruler for this.) Then, you'll follow the rules below to find out how many hits it takes for the ball to make it to a corner pocket and which pocket the ball ends up going in. You are looking for PATTERNS.

The Rules:

- Every table has a pocket in each of its four corners.
- Start in the lower left hand corner every time.
- The ball always moves at a 45 degree angle until it either hits the side of the table or lands in a corner pocket.
- When the ball hits the side of the table it bounces at a 90 degree angle.
- We keep following the path of the ball until it lands in a corner pocket.

What to do:

1. Using graph paper, draw a rectangular pool table that has the length (↔) to width (↕) ratio given.
2. Use the rules above to follow the path of the ball until it lands in one corner of the table.
3. Record the number of hits and the pocket it landed in in the data table below. (The ball leaving the lower left corner counts as the first hit and the ball landing in the pocket counts as the last hit. The other hits will come from bouncing off the sides of the table.)
4. Repeat this process for all of the ratios in the data table.
5. When your table is complete, answer the questions on page 2 to try to draw some conclusions.

Length to Width Ratio	**Write the ratio as a REDUCED fraction…**	**Number of hits**	**Pocket it landed in (upper right, lower right, upper left, or lower left)**	**Length to Width Ratio**	**Write the ratio as a REDUCED fraction…**	**Number of hits**	**Pocket it landed in (upper right, lower right, upper left, or lower left)**
1:1				3:1			
1:2				3:2			
1:3				3:3			
1:4				3:4			
1:5				3:5			
1:6				3:6			
1:7				3:7			
1:8				3:8			
2:1				4:1			
2:2				4:2			
2:3				4:3			
2:4				4:4			
2:5				4:5			
2:6				4:6			
2:7				4:7			
2:8				4:8			

Conclusions:

Look at the reduced ratios and the number of hits it took before the ball landed in a corner. What relationship do you see between the reduced ratio and the total number of hits? Be specific.

Look carefully at the reduced ratios and the pocket that the ball ultimately landed in. What relationship do you see between the reduced ratio and the pocket the ball goes in? Be specific.

Discuss your conclusions with your class. Can you and your classmates come up with a set of rules to determine how we might predict the path of a ball ONLY by knowing the dimensions of the table? After your discussion, write your set of rules here.

<u>Making Predictions</u>

Based on the rules you and your classmates came up with, you are now going to make predictions about the path of a ball based on the dimensions of the table. (Remember, length is ↔ and width is ↕.)

1. A table is 26 feet wide and 12 feet long.
    a. Write the length to width ratio for this table in reduced form.

    b. Use your ratio from part a to predict (1) how many hits before the ball goes in a pocket and (2) what pocket the ball will finally go in. Explain your answers.

    c. Draw the table and the path of the ball on graph paper. Does this math your prediction? If not, explain the problem.

2. A table is 25 feet long and 35 feet wide.
    a. Write the length to width ratio for this table in reduced form.

    b. Use your ratio from part a to predict (1) how many hits before the ball goes in a pocket and (2) what pocket the ball will finally go in. Explain your answers.

    c. Draw the table and the path of the ball on graph paper. Does this math your prediction? If not, explain the problem.

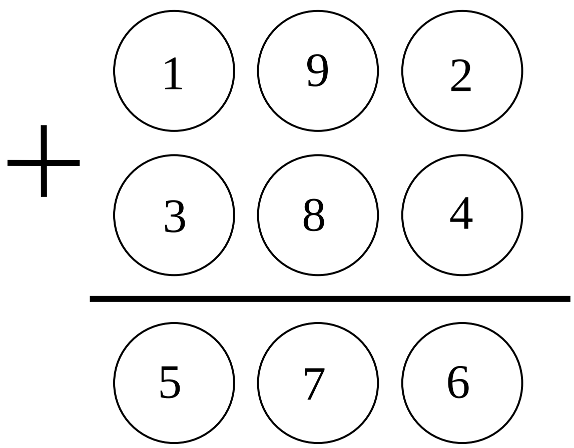
3. A table is 1080 feet long and 240 feet wide. Predict the path of the ball.

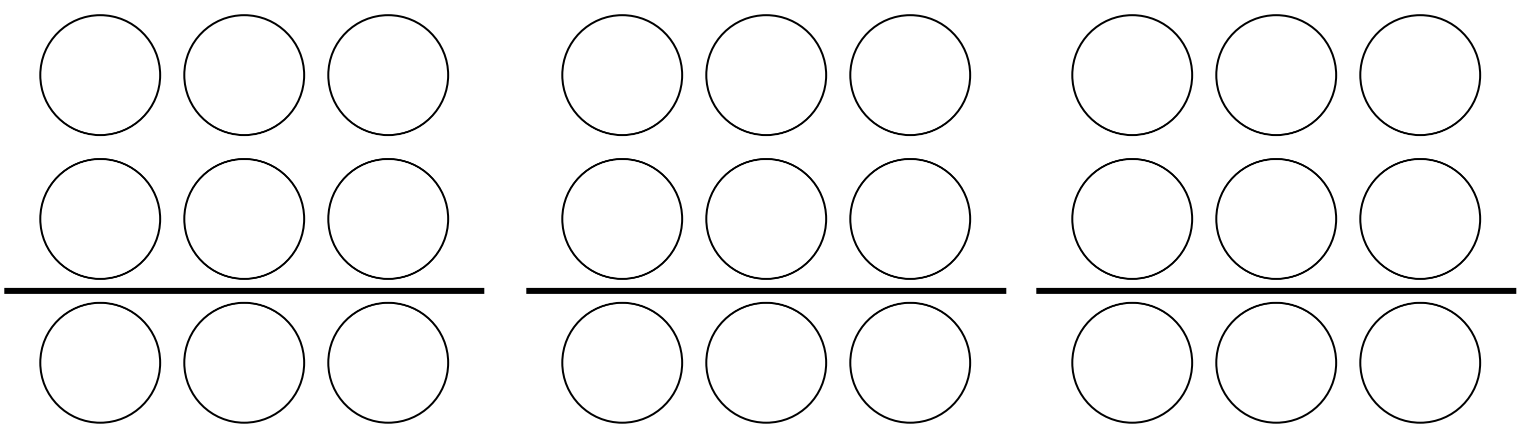
4. A table is 72 feet long and 432 feet wide. Predict the path of the ball.

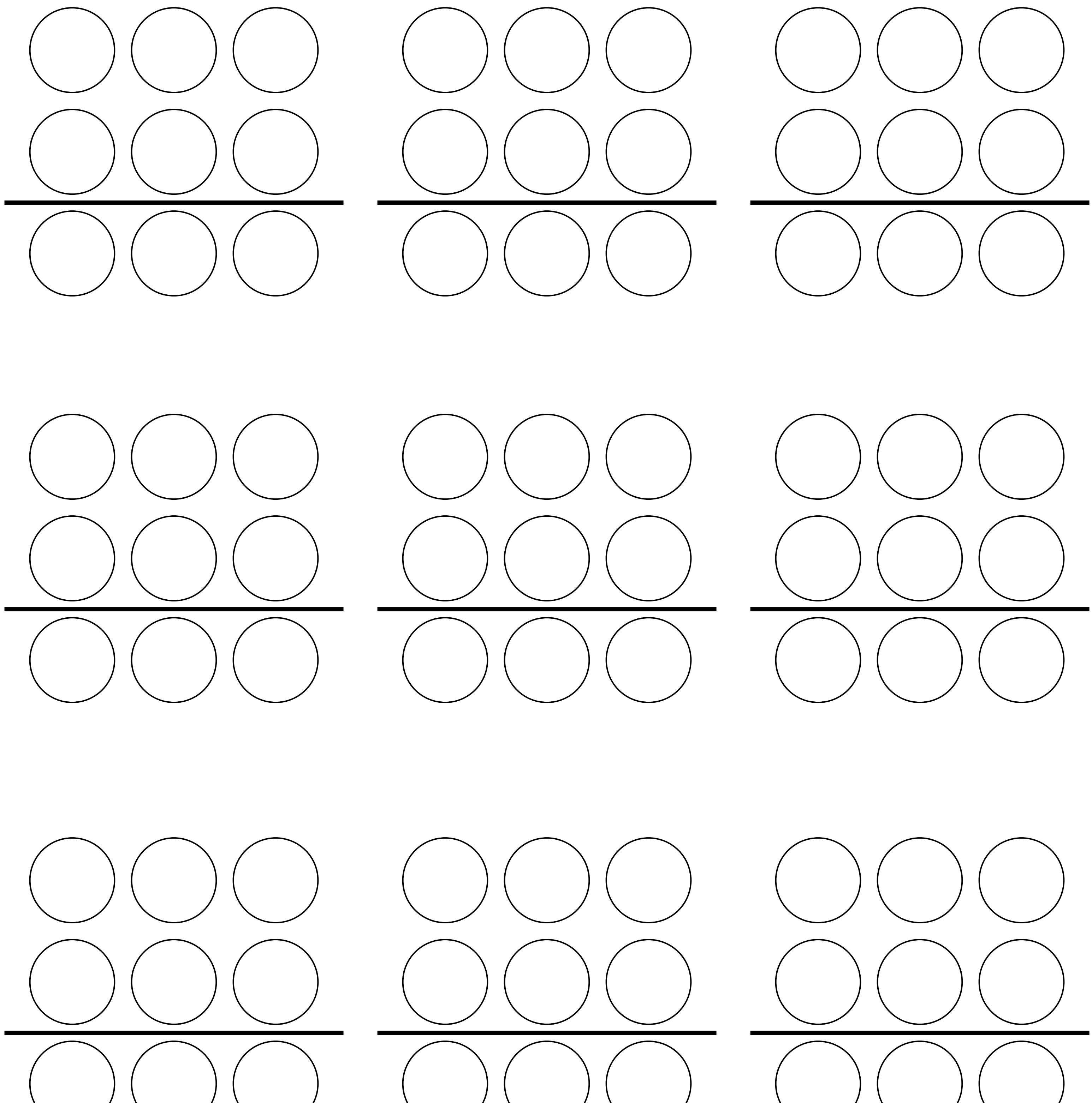
Appendix B

Number Plays

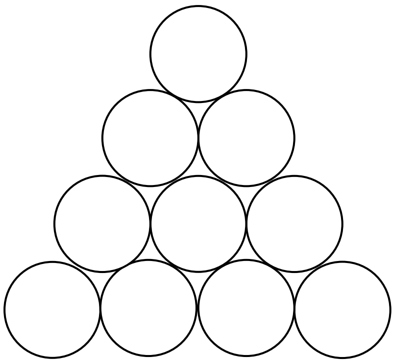
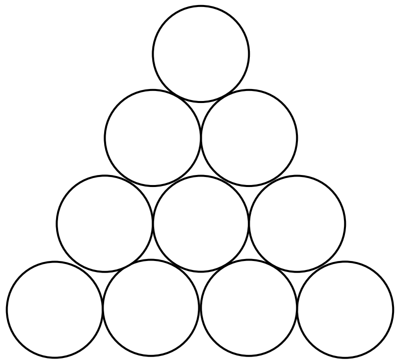
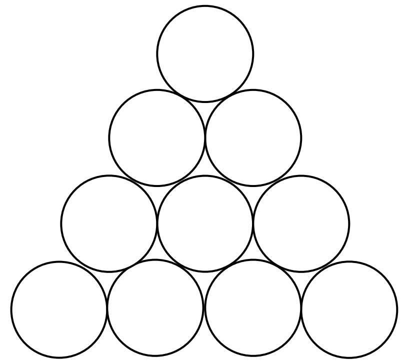
Place the numbers 1-9 in the circles so that they show a true addition problem (just like in the example). Find as many different solutions as you can. Are there any patterns that you notice that help you to find more solutions?

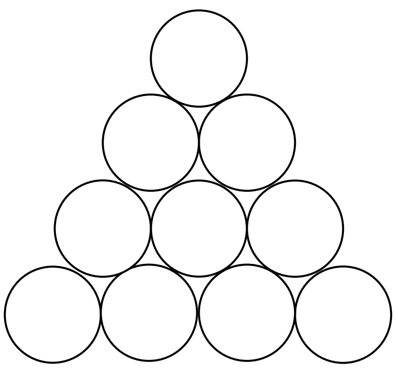
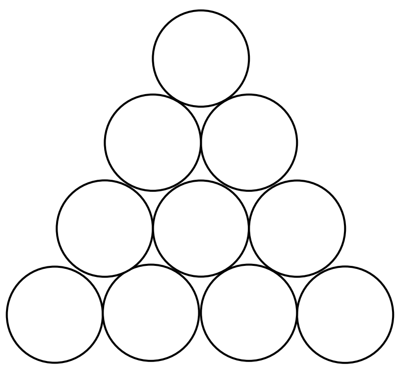
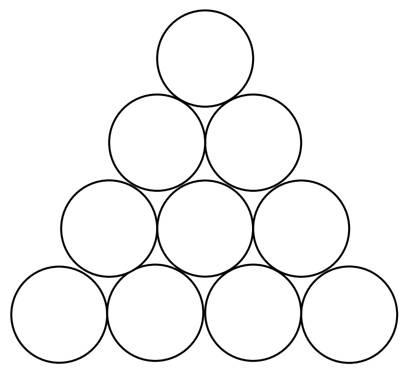


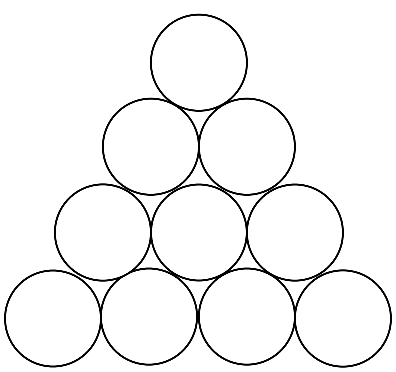
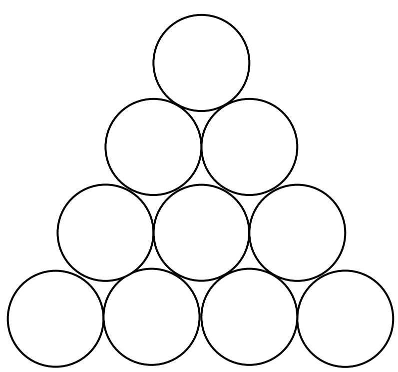
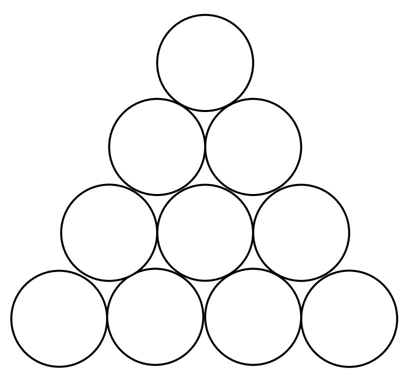


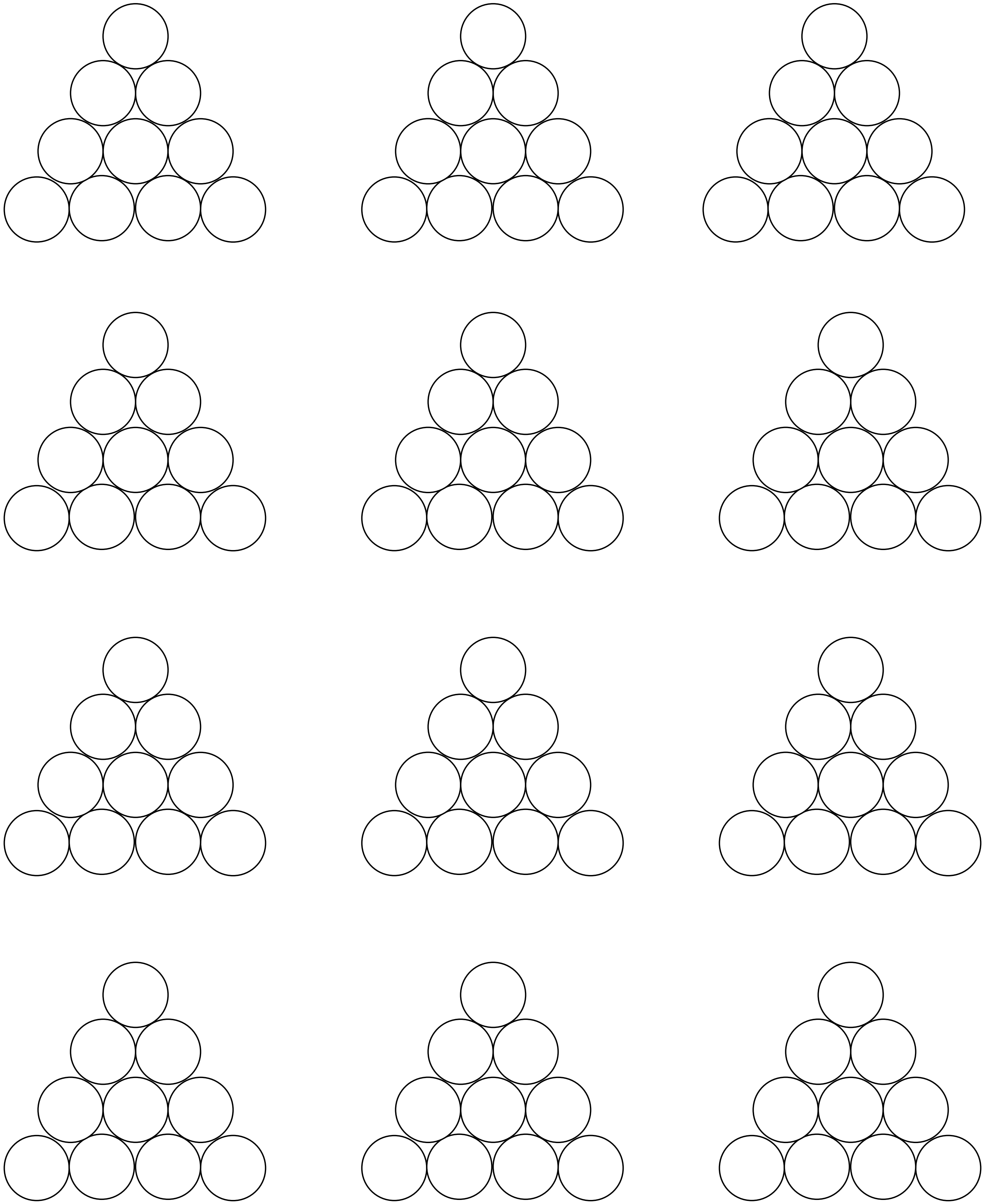


Place the numbers 1-10 in the circle below so that the number in each circle is the sum of the numbers in the circles below it. Find as many different solutions as you can. Are there any patterns that you notice that help you to find more solutions?



Appendix C

A's and B's Are Numbers Too

1. Adding Fractions: Find the values of A and B below.

$$\frac{1}{3} = \frac{1}{A} + \frac{1}{B}$$

2. Multiplication: In the multiplication problem below, A and B stand for different digits. What are the values of A and B?

```
        A B
      x B A
     ------
    1 1 4
  3 0 4
  -------
  3 1 5 4
```

3. Multiplication: In the multiplication problem below, each blank space represents a missing digit. Find the product and the original multipliers.

```
      4 _ _
    x   _ 7
    _ _ 8 2
  ---------
  1 2 _ _
  _ _ _ _ _
  ---------
```

Appendix D

Teacher and Student Resources

National Council for Teachers of Mathematics: Principles and Standards for School Mathematics Process Standards http://standards.nctm.org/document/appendix/process.htm

The mathematics of Billiards on rectangular, circular, elliptical, triangular, and other types of "billiard" tables http://mathworld.wolfram.com/Billiards.html

More on elliptical (and circular) billiard tables http://cage.rug.ac.be/~hs/billiards/billiards.html

The clip of Donald Duck in Mathemagicland can be found online at http://youtube.com/watch?v=qLTKX4WxmZ4&feature=related