

# Playing with Primes

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## Overview

This unit utilizes a series of individual and small group explorations, combined with class discussions and presentations, to expand students' understanding of prime numbers and other aspects of number theory. Going beyond traditional factor-tree prime factorization, it focuses on patterns within the primes, their history, their usefulness in fractions, in divisibility and in problem solving, and other aspects of their "specialness."

## Rationale

As a middle school mathematics teacher, teaching in a variety of settings, using a wide variety of curricula, I have always been aware that many students think of mathematics as something tedious, or even worse, frightening. They seem to feel that there are secrets about being good at math that they will never learn, and so become convinced that they will never be good at it. This attitude seems to develop quite early on in their educational careers, and persists throughout high school. Interestingly, it does not directly correspond to overall intelligence or success in other academic areas. Many excellent students have a blind spot when it comes to mathematics, and seem to be perfectly content to say they are bad at math.

As a teacher in the Pittsburgh Public Schools Gifted program, teaching gifted sixth graders from throughout the city, including private and parochial schools, I am in a unique position to see what can be done about this. Some of the students truly enjoy mathematics and are extremely gifted in all of its aspects, the majority are competent at math, and a few even have "the attitude." Overall, however, too many seem to think that math is "hard" and "boring." Since all of these students have the potential for succeeding academically, it would seem that material presented in an accessible and interesting way might open them to the intriguing aspects of mathematics and make it seem less tedious.

Prime numbers are one of those intriguing areas of mathematics. They are the atoms of mathematics, the building blocks from which all other numbers are made. But they have always been mysterious. From Euclid's Proposition 20, in which he explains his theory that there are infinitely many prime numbers, through Fermat and Goldbach and Euler and Gauss (plus many others), to the

present time of RSA encryption, mathematicians have always been fascinated by the primes. Mathematicians' obsession with structure, logic and "provability" has made the proofs of conjectures and hypotheses about primes the holy grails of theoretical mathematics. Exposing students to the richness and fascinating history of prime number theory should open new insights into areas of math that go beyond typically dry arithmetic.

One of the best things about working with prime numbers is their accessibility. They are easily defined, a manageable size (at least for our purposes), and most middle school students have been exposed to them in elementary school. In sixth grade, a unit on primes is taught that coalesces informal understanding into the formal understanding of number theory. So the typical sixth grade student will know the definition of a prime number, will know about prime factorization, will have some understanding of its use in finding least common multiple and greatest common factor and will have some experience in solving problems related to prime numbers. At this point, all the tools are in place for a deeper exploration of the primes and their history.

The middle school mathematics curriculum for the Pittsburgh Public Schools is *Connected Mathematics*. It is based on the Launch-Explore-Summarize model, which allows students to investigate problems in order to develop mathematical understanding. The activities in *Playing with Primes* follow this model and may be used at any middle school level, either as reinforcement within the curriculum, enrichment, or projects, either individual or group.

Each Exploration looks at different facets of prime numbers and number theory. Exploration 1 uses the Sieve of Eratosthenes to find the primes up through 500. Once the students have completed the sieve and made their list, they can look for patterns both visually and numerically. The completed sieve is used throughout the rest of the explorations.

Exploration 2 asks the students to find pairs of twin primes and pairs of relatively prime numbers. The twin primes are simple to find, using either the sieve or the list from Exploration 1. Finding relatively prime numbers involves searching among the composite numbers up to 500 for those with no common factors other than 1. This is rather complicated since there are many possible pairs, and gives the students experience in organizing data. Searching for relatively prime pairs reinforces students' understanding and knowledge of factors and helps them to see the composite numbers as groups of multiples.

Exploration 3 presents a list of divisibility tests to use in deciding if large numbers are prime. Some of these tests should already be familiar to the students, others, such as the tests for 7 and 11, are quite exotic. As the students use the

tests on their numbers, they will begin to develop an understanding of what divisibility means, as well as learning some useful testing skills.

Exploration 4 uses Goldbach's Conjecture as a starting point. After some historical background, students will test his conjecture by finding prime sums for all the numbers from 4 to 99. The even ones, he conjectured, could all be written as the sum of two primes, the odds as the sum of three. Organization is important in this exploration, as well as the pattern observation skills that have been used in the others.

In Exploration 5, three classifications of numbers are introduced – perfect, abundant and deficient. After defining the types, each student will list all the factors of the assigned numbers and classify the numbers as perfect, the sum of all factors other than the number itself is equal to the number; abundant, the sum is greater than the number; or deficient, the sum is less than the number. Once all the assigned numbers have been grouped, the students will look for similarities within the groups and differences among them. This activity reinforces understanding and knowledge of factors and helps students develop facility in listing factor pairs. It also introduces them to an interesting formula for finding perfect numbers. Students at this level are not familiar with the use of formulas, and this is an interesting and practical introduction to patterns and formulae.

Exploration 6 is a practical one. It asks the students to find prime factorizations by using factor trees. As each unique prime factorization is found and entered in the class chart, patterns and groupings of numbers can be observed. The students keep their own lists of the factorizations which they will use to solve the problems in Exploration 8.

A historical topic is introduced in Exploration 7. Aliquot fractions, those with numerators of 1, were the only way fractions were written by the Egyptians. To write fractions other than one part of a whole, they added together two or more aliquot fractions. This exploration gives students practice in common denominators and fraction addition.

Exploration 8 gives the students a chance to use the skills they have learned and the lists they have generated to solve problems. The problems are varied, dealing with many aspects of number theory and should give the students experience both in problem solving and in organization.

For classrooms with access to the internet, students may explore prime number topics. Using any search engine, they can enter "Prime Numbers" and get more than enough references to write a short essay on a topic related to prime

numbers and number theory. The assignment should be left as open as possible to allow freedom for exploration and individual interests.

As a summary activity after each exploration, students will complete a journal entry about the exploration. This will solidify the mathematics of the exploration and give them practice in writing about mathematical topics.

### **Objectives**

There are several objectives for this unit. The most obvious is to increase students' understanding of the properties of prime numbers. A second objective is to teach students ways in which prime number theory can be used in other areas of mathematics. Thirdly, as students work on the various explorations of this unit, they will practice their calculation skills and concepts of divisibility, touch upon areas of probability and statistics, and expand their problem solving strategies. Finally, the ongoing research element will integrate research and writing skills as it develops an awareness of mathematics as history.

All eleven mathematical standards will be addressed in this unit, calculation and number theory ones more so than those dealing with geometry and measurement.

### **Strategies**

Each exploration will be presented to the class with historical background where appropriate. After the problem is presented and procedures explained as necessary, students will work, depending on the exploration, either individually, in small groups, or as a class to complete the task. Near the end of the allotted time, class discussion will reinforce the mathematics of the exploration.

As students are working, the inquiry and experimentation nature of the explorations should be reinforced. When possible, students should work together to answer their own questions and get through their difficulties independently. The teacher's role should be one of advisor rather than problem-solver.

Students will make journal entries after each exploration to solidify their understanding of the mathematics. Any format may be used as long as the entries discuss the mathematical concepts of the exploration.

### **Format**

The explorations are formatted so that the student pages are separate from the teacher tips, etc., allowing them to be copied and used as is. There are no specific

directions for setting up the class charts. They may be done on the overhead, chalk board or on chart paper. If possible, they should be saved as reference material for use throughout the unit.

### **Summaries of the Explorations**

#### Exploration 1: Finding Primes and Looking for Patterns

Students will use the Sieve of Eratosthenes to find the prime numbers up to 500. They will then list them, graph the numbers of primes and describe any patterns they see.

#### Exploration 2: Twin Primes and Relatively Prime Numbers

Looking at the list generated from Exploration 1, students will find pairs of twin primes. They will look for numbers that are relatively prime and generate a list to be entered in a class chart.

#### Exploration 3: Deciding if a Number is Prime

Each student will be assigned five large numbers and will try different strategies to decide if they are prime or composite. They will enter their numbers into a class chart to look for patterns.

#### Exploration 4: Sums of Primes

Students will work in groups to write numbers from 4 through 99 as the sums of two or three primes. As they find a solution for each number, they will enter it in a class chart.

#### Exploration 5: Perfect, Abundant and Deficient Numbers

Each student will be given five numbers. They will work individually to classify each number as Perfect, Abundant or Deficient. Results will be combined and common properties of each type will be discussed.

#### Exploration 6: Factor Trees and unique Prime Factorization

Students will find the prime factorizations of numbers. They will keep an organized list of the factor strings which will be used in problem solving.

#### Exploration 7: Aliquot Fractions

Aliquot fractions are introduced and defined. Students will work in small groups to rewrite fractions in Aliquot form. A class chart will be made of the results.

#### Exploration 8: Solving Problems with Primes

Students will be given problems that can be solved by the number theory

skills they have used in the previous explorations. Using the results of the explorations as resources, they will work with their group to solve each problem and write detailed solutions.

## Exploration 1: Finding Primes & Looking for Patterns

### The Sieve of Eratosthenes

#### History

Eratosthenes was an all-round scholar, born in Cyrene (now Libya) in 276 BC. Although he wrote in many areas, was the first to estimate the diameter of the Earth, and served as the librarian of the Library at Alexandria for many years. he is probably best remembered for his method of finding prime numbers, known as the Sieve of Eratosthenes.

#### Directions for this Exploration

Step 1: Working with your group, use the sieve to find the prime numbers up to 500.

Step 2: Make a list of the primes. Check your list with another group to see if it matches. If not, figure out who is right.

Step 3: Complete a coordinate graph that shows the number of primes.

Step 4: Looking at your list and your graph, describe any patterns you see.

#### How to use the Sieve

Step 1: Cross out the number 1 – it is neither prime nor composite.

Step 2: Circle 2 – it is a prime number.

Step 3: Using a highlighter, highlight all the multiples of 2.

Step 4: Circle 3 – that is the next prime number.

Step 5: Highlight all the multiples of 3 that are not already highlighted.

Step 6: Circle 5, the next unhighlighted number – it is prime.

Step 7: Highlight all the multiples of 5 that are not already highlighted.

Continue doing this until you have either circled or highlighted all the numbers. The circled numbers are the Primes. The highlighted numbers are the Composites.

### Making the Coordinate Graph

The coordinate pair for each point will be (numbers so far, number of primes). For example – counting up, at 1, there are 0 primes; at 2, there is one prime; at 3 there are two primes; at 4, there are 2 primes; at 5, there are 3 primes, etc. The first five points to be graphed, then, are (1,0); (2,1); (3,2); (4,2); (5,3).

Step 1: Set up your x-axis so that it goes from 0 to 200.

You may need to use more than one piece of paper for this. If so, just continue the axis from where you left off on the first paper.

Step 2: Set up your y- axis so that it goes from 1 to 50.

Step 3: Write the coordinate pairs for each number – these are the points you will graph.

Step 4: Carefully graph each point. Do not connect the points.

INSERT GRAPH HERE

## Discussion Questions

- When you were using the Sieve of Eratosthenes to find the primes, what shortcuts did you discover that helped you?
- Were there any visual patterns in the sieve?
  
- Do you see any patterns in your list of primes?
- Which is the most frequent ones digit?
  
- Describe the pattern in the graph.

## Teacher Tips

As the students get into the higher numbers, they can use a calculator to find the multiples of the numbers. You can lead them to the technique of only multiplying the base prime number (the one they have just circled) by the primes that are larger, since all the other multiples will already have been crossed out.

They will know they are done when their base prime number multiplied by the next biggest prime is over 500. Everything left at that point is prime.

They will definitely need help with the graph, both in setting up the axes and graphing the points.

## Exploration 2: Twin Primes & Relatively Prime Numbers

### Definitions

**Twin Primes** are two prime numbers that are one number apart.

For example, 17 and 19 are twin primes, since they are separated only by 16.

**Relatively Prime Numbers** are composite numbers that have no common factors other than 1.

For example, 15 and 16, both composite, have only 1 as a common factor.

### Directions for this Exploration

#### Part 1 – Twin Primes

Step 1: Using your list of primes and/or your Sieve of Eratosthenes, make a list of all the sets of twin primes up to 500.

Step 2: Look at your list. Write down any patterns you see.

#### Part 2 – Relatively Prime Numbers

Step 1: Using your Sieve of Eratosthenes, look for pairs of relatively prime numbers.

Step 2: As you find each one, send someone from your group up to enter the pair on the class chart.

Step 3: As the chart fills up, look for patterns and relationships – we will discuss them as a class.

## Discussion Questions

- How often do the twin primes occur?
- Is there anything special about the number that separates them?
- Is there anything special about the pairs themselves?
- The difference of the twins is always 2 – is there a pattern in the sums?
  
- What strategies did you use to find the relatively prime numbers?
- Which pair is the closest together?
- Which pair is the farthest apart?
- What made a pair easy or hard to find?
- Are some numbers used more than others?

## Teacher Tips

The introductory conversation for finding relatively prime numbers should point out obvious places to look for the pairs and obvious places not to look. All even numbers, for example, are divisible by 2, so no pairs of evens would be relatively prime. Good places to look for one half of the pair would be within the multiples of 3, 7, 11, or other primes, and look for the other half in the even numbers.

## Exploration 3: Deciding if a Number is Prime

### Directions for this Exploration

You will be given five large numbers. Use any methods you like to decide if the numbers are prime or composite. When you find a prime number, enter it on the class chart.

The following chart will help.

### Divisibility Tests

A number is divisible by...	if...
2	the last digit is an even number
3	the sum of the digits is divisible by 3
4	the last two digits are divisible by 4
5	the last digit is 5 or 0
6	the number is divisible by 2 and 3
7	the sum of the number formed by the last six digits and the number formed by the rest of the digits is divisible by 7
8	the last three digits are divisible by 8
9	the sum of the digits is divisible by 9
10	the number ends in 0
11	the difference between the sum of the odd numbered digits (1st, 3rd, 5th...) and the sum of the even numbered digits (2nd, 4th...) is divisible by 11
12	the number is divisible by 3 and 4
13	the sum of the number formed by the last six digits and the number formed by the rest of the digits is divisible by 13
Other numbers	it is divisible by all of the other number's factors

Hint: when you are using your calculator to check, you only need to divide by prime numbers (use your list or the sieve to find them). When your quotient is smaller than your divisor, you can stop.

### Discussion Questions

- What strategies did you use?
- Which numbers can you just look at to see what they are divisible by?
- Do the prime numbers have any common traits?

### Some Numbers to Use

All of the composite numbers have at least one factor less than 20. Quite a few have 3, 7, 11 or 13, so those divisibility tests can be used. Try to encourage the students to use the divisibility tests.

### Composites

1,089 1,683 3,757 3,887 4,669 4,807 5,423 5,529 6,859 9,945  
10,933 11,991 12,261 12,711 13,433 14,443 18,139 19,019 20,907  
25,553 26,163 26,809 29,391 32,623 33,229 37,927 41,173 44,033  
46,189 46,607 49,283 53,298 63,019 63,631 72,347 73,271 84,021  
86,367 94,329 107,767 108,851 136,317 150,449 151,863 157,661  
161,051 166,141 174,229 186,143 197,417 199,139 204,493 247,753  
275,329 281,203 318,041 359,579 367,327 371,293 374,407 496,463  
530,347 676,359 678,923 975,711 1,067,803 1,294,069 1,640,989  
4,567,321 7,235,423 9,959,323 11,183,227 12,623,809 68,312,259  
70,022,229 120,311,259

### Primes

1,013 1,381 1,399 2,111 2,377 2,837 2,579 3,191 3,539 3,739  
3,779 3,797 4,051 4,091 4,621 4,663 4,889 4,987 5,039 5,387  
6,277 6,451 6,863 7,879 8,443 8,663 8,923 9,091 10,457 11,149  
12,721 13,751 14,323 15,919 16,363 17,981 18,701 19,801 20,593  
21,851 22,921 23,899 24,533 25,229 26,407 27,179 28,813 29,059  
30,649 31,727 32,653 33,119 34,819 35,141 36,263 37,189 38,933  
39,847 40,093 41,051 42,773 43,049 44,483 45,697 46,229 47,809  
48,989 95,231 96,799 99,709 100,333 101,449 102,877 103,843  
104,717

## Generating Numbers

The easiest thing to do is multiply two or more large primes together, then multiply that product by a prime under 20. Large prime numbers can be found on the internet at many sites.

## Exploration 4: Sums of Primes

### History

Christian Goldbach (1690- 1764) was a Russian mathematician, living in St. Petersburg and, later, Moscow. He is mostly remembered for his conjecture, still unproven, that any even number greater than 2 can be written as the sum of two primes. He also conjectured that any odd number can be written as the sum of three primes.

### Directions for this Exploration

You will work as a class to write all the numbers from 4 through 99 as the sums of primes. Use two primes for the evens and three for the odds. As you find each new sum, write it on the table below and come up to enter it on the class chart. Watch as other students enter their numbers, and copy those into the table too so you don't repeat work that has already been done.

No.	Prime Sum	No.	Prime Sum	No.	Prime Sum	No.	Prime Sum
4		28		52		76	
5		29		53		77	
6		30		54		78	
7		31		55		79	
8		32		56		80	
9		33		57		81	
10		34		58		82	
11		35		59		83	
12		36		60		84	
13		37		61		85	
14		38		62		86	
15		39		63		87	
16		40		64		88	
17		41		65		89	
18		42		66		90	
19		43		67		91	
20		44		68		92	
21		45		69		93	
22		46		70		94	
23		47		71		95	

24		48		72		96	
25		49		73		97	
26		50		74		98	
27		51		75		99	

### Discussion Questions

- Which were the easiest numbers to write as sums of primes?
- What patterns for finding the sums did you discover?
- Can some numbers be written in more than one way?

### Teacher Tips

This is an easily understandable exploration. Remind the students to use their Sieve of Eratosthenes as a resource. Also remind them to check the chart frequently, not only to see which sums have been found, but also to see if any patterns are developing that could be helpful to them.

## Exploration 5: Perfect, Abundant & Deficient Numbers

### Definitions

A number is **Perfect** if it is equal to the sum of its factors, excluding itself.

A number is **Abundant** if it is less than the sum of its factors, excluding itself.

A number is **Deficient** if it is more than the sum of its factors, excluding itself.

### Directions for this Exploration

Before you begin working on this exploration, look at the definitions of Perfect, Deficient and Abundant numbers. Write down your prediction of which will be the most common type of number.

You will be assigned 5 numbers to explore. For each of your assigned numbers, list all of the factors. It is easiest if you do this in pairs, for example, the factors of 12 are  $1 \cdot 12$ ,  $2 \cdot 6$  and  $3 \cdot 4$ . Next, add together the factors, except for the number itself. The sum will tell you what kind of number it is, according to the definitions above. Enter your number in the proper category on the class chart. As the chart fills up, look for common characteristics of the numbers in each category.

### Finding Perfect Numbers

There is a formula for finding perfect numbers. Follow the directions below to find the first 5 perfect numbers.

Start with 1.

Add on 2. Did you get a prime number? (yes – 3)

Multiply that by 2. ( $3 \times 2 = 6$ )

6 is a Perfect Number. ( $1 + 2 + 3 = 6$ )

Now add on 4 ( $2^2$ ).

Did you get a prime number? (yes –  $1 + 2 + 4 = 7$ )

Multiply that by 4, the last number you added on. ( $7 \times 4 = 28$ )

28 is a Perfect Number. ( $1 + 2 + 4 + 7 + 14 = 28$ )

Keep adding on powers of 2. The next one is 8, then 16, then 32, etc.

If you get a composite number, go on to adding the next power of 2. When you get to a prime number, multiply it by the last power of 2 that you added on. That product will be a perfect number. Use your sieve to check for primes. You will need a calculator – the numbers get big very quickly.

### Discussion Questions

- Was your prediction right?
- Is there some type of number that is always deficient?
- What are some common characteristics of Abundant numbers?

### Numbers to Use

Here is a list of some possible numbers, their factors and their classifications. You will want to add in some primes, and whatever other numbers you think might be interesting.

4 – 1, 2, 4 - Deficient  
6 – 1, 2, 3, 6 - Perfect  
8 – 1, 2, 4, 8 - Deficient  
10 – 1, 2, 5, 10 - Deficient  
15 – 1, 3, 5, 15 - Deficient  
16 – 1, 2, 4, 8, 16 - Deficient  
20 – 1, 2, 4, 5, 10, 20 - Abundant  
24 – 1, 2, 3, 4, 6, 8, 12, 24 - Abundant  
25 – 1, 5, 25 - Deficient  
28 – 1, 2, 4, 7, 14, 28 - Perfect  
30 – 1, 2, 3, 5, 6, 10, 15, 30 - Abundant  
36 – 1, 2, 3, 4, 6, 9, 12, 18, 36 - Abundant  
40 – 1, 2, 4, 5, 8, 10, 20, 40 - Abundant  
45 – 1, 3, 5, 9, 15, 45 - Deficient  
48 – 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 - Abundant  
50 – 1, 2, 5, 10, 25, 50 - Deficient  
60 – 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 - Abundant  
64 – 1, 2, 4, 8, 16, 32, 64 - Deficient  
70 – 1, 2, 5, 7, 10, 14, 35, 70 - Abundant  
72 – 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72 - Abundant  
75 – 1, 3, 5, 15, 25, 75 - Deficient  
80 – 1, 2, 4, 5, 8, 10, 16, 20, 40, 80 - Abundant  
85 – 1, 5, 17, 85 - Deficient  
88 – 1, 2, 4, 8, 11, 22, 44, 88 - Abundant  
90 – 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90 - Abundant  
96 – 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96 - Abundant

98 – 1, 2, 7, 14, 49, 98 - Deficient

100 - 1, 2, 4, 5, 10, 20, 25, 50, 100 - Abundant

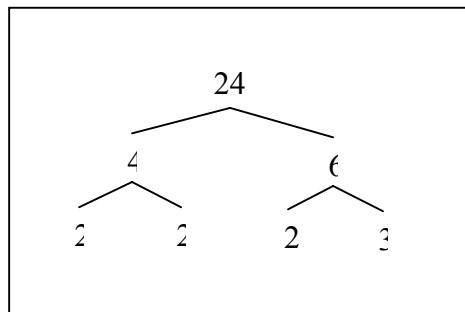
### Teacher Tips

As the students are finding factors, suggest that they look at the pairs as they go along. They will be able to find factors from other factors. For example, once they know that 98 has the pair 2 and 49, they will know that 7 goes into it.

## Exploration 6: Factor Trees and Unique Prime Factorizations

### Directions for this Exploration

In this exploration, you will be finding the unique prime factorizations for composite numbers. Below is an example of a factor tree.



The ends of the branches, when multiplied together, equal the starting number.

$$2 \times 2 = 4 \times 2 = 8 \times 3 = 24,$$

so the prime factorization of 24 is  $2 \times 2 \times 2 \times 3$ .

This can also be written as  $2^3 \times 3$

For each of your assigned numbers, use a factor tree to find the prime factorization. You might want to look back at the divisibility rules in Exploration 3 to help get started with some of the harder numbers.

As you finish, enter the prime factorization for each number in the class chart.

When all the numbers have been entered in the chart, look at them with your group and talk about these questions:

1. Do some numbers seem to “go together” – meaning they are made up of mostly the same prime numbers? If so, make a list of those numbers.
2. Are some of the numbers “special” – meaning their prime factorizations all have the same pattern? Make a list of those numbers.
3. Is there any one prime factor that is in all or most of the numbers?
4. Can you tell by looking at the number before you factor it what prime factors it will have? What clues are there?

## Teacher Tips

Students will probably need some help getting started with the factor trees. Doing a few together as a class should be enough.

You can use any numbers. Below is a list grouped into “families”. Some numbers are in more than one family. If you assign each student numbers from different families, they can look for similarities and differences as they are entered into the class chart. This is by no means meant to be a definitive list.

As the students are discussing patterns they see in the prime factorizations, be sure they notice that square numbers have equal pairs and cube numbers have sets of three.

## Numbers to Use

powers of two: 4 8 16 32 64 128 256 512 1,024

powers of three: 9 27 81 243 729 2,187 6,561 19,683 59,049

twos and threes: 12 18 24 30 36 48 54 60 72 84 90 96

powers of five: 25 125 625 3,125 15,625 78,125 390,625 1,953,125  
9,765,625

fives and sevens: 14 15 20 21 25 28 30 35 40 42 45 49 50 55 56 60 63  
65 70 72 77 80 84 85 90 91 95

squares: 4 9 16 25 36 49 64 81 100 121 144 169 196 225

cubes: 8 27 64 125 216 343 512 729 1,000

elevens and thirteens: 22 26 33 39 44 52 55 65 66 77 78 88 91 99

seventeens and nineteens: 34 38 51 57 68 76 85 95 102 114

hard: 143 187 209 221 247 253 299 323 391 437

## Exploration 7: Aliquot Fractions

### History and Explanation

In Egypt, written fractions had only numerators of 1, like  $\frac{1}{3}$  or  $\frac{1}{7}$ . These are called aliquot fractions. When they needed to write other fractions, such as  $\frac{5}{6}$ , they would write it as the sum of two or more aliquot fractions, in this case, as  $\frac{1}{2} + \frac{1}{3}$ . You can prove that  $\frac{1}{2} + \frac{1}{3}$  is the same as  $\frac{5}{6}$  by changing to a common denominator and adding:  $\frac{1}{2} + \frac{1}{3}$  is  $\frac{3}{6} + \frac{2}{6}$  which is  $\frac{5}{6}$ . When writing a fraction as the sum of aliquot fractions, each fraction could only be used once, so  $\frac{5}{6}$  would not be written as  $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ .

### Directions for this Investigation

Working with your group, you will write fractions as sums of aliquot fractions. All the fractions you will be given can be written that way, but some may take more guessing and checking than others. Some fractions may have more than one answer.

Here are some hints:

Only factors of the denominator can be used as numerators.

Example: for a denominator of 12, you could use  $\frac{2}{12}$  ( $\frac{1}{6}$ ),  $\frac{3}{12}$  ( $\frac{1}{4}$ ),  $\frac{4}{12}$  ( $\frac{1}{3}$ ) and  $\frac{6}{12}$  ( $\frac{1}{2}$ ). This means you could write  $\frac{5}{12}$  as  $\frac{1}{6} + \frac{1}{4}$ , or  $\frac{7}{12}$  as  $\frac{1}{4} + \frac{1}{3}$ .

Start by seeing if you can use the given denominator. If not, multiply the numerator and denominator by 2. If that doesn't work, try multiplying by other numbers until you find an equivalent fraction that can be broken into aliquot parts.

Example:  $\frac{3}{5}$  could only be  $\frac{1}{5} + \frac{2}{5}$ , which doesn't work.

$\frac{6}{10}$  is equivalent to  $\frac{3}{5}$ . It can be written as  $\frac{1}{10} + \frac{5}{10}$  and the  $\frac{5}{10}$  is equivalent to  $\frac{1}{2}$ . So the aliquot fraction sum for  $\frac{3}{5}$  is  $\frac{1}{10} + \frac{1}{2}$ .

Once you have decided on a denominator, it might save time to write all possible fractions that reduce to aliquots.

Example: if the denominator you decide to try is 8, then the fractions you could use would be  $\frac{1}{8}$ ,  $\frac{2}{8} = \frac{1}{4}$ , and  $\frac{4}{8} = \frac{1}{2}$ .

As you find the aliquot sums for your fractions, write them on the class chart.

## Teacher Tips

This is complicated, so the students will need to work through a few examples as a class. An alternative, and easier, activity would be to add up different aliquot fractions to see what sums you can get.

## Fractions to Use

These are grouped so they can be presented as sets and the students can be told which denominators to try. Many of them work with more than one denominator, and many have more than one solution. As you assign the fractions, each group should get a few obviously easy ones.

An interesting task would be to assign the class a fraction that appears frequently, such as  $\frac{3}{4}$ , and have them find the sums using as many different denominators as possible.

Using 24 as a denominator:  $\frac{5}{24}$   $\frac{7}{24}$   $\frac{3}{8}$   $\frac{5}{12}$   $\frac{11}{24}$   $\frac{13}{24}$   $\frac{7}{12}$   $\frac{5}{8}$   $\frac{2}{3}$   
 $\frac{17}{24}$   $\frac{3}{4}$   $\frac{19}{24}$   $\frac{5}{6}$   $\frac{7}{8}$   $\frac{11}{24}$   $\frac{23}{24}$

Using 18 as a denominator:  $\frac{2}{9}$   $\frac{5}{18}$   $\frac{7}{18}$   $\frac{4}{9}$   $\frac{5}{9}$   $\frac{11}{18}$   $\frac{2}{3}$   $\frac{13}{18}$   $\frac{7}{9}$   $\frac{5}{6}$   
 $\frac{8}{9}$   $\frac{17}{18}$

Using 20 as a denominator:  $\frac{3}{20}$   $\frac{3}{10}$   $\frac{7}{20}$   $\frac{2}{5}$   $\frac{9}{20}$   $\frac{11}{20}$   $\frac{3}{5}$   $\frac{13}{20}$   $\frac{7}{10}$   
 $\frac{3}{4}$   $\frac{4}{5}$   $\frac{17}{20}$   $\frac{9}{10}$   $\frac{19}{20}$

Using 16 as a denominator:  $\frac{3}{16}$   $\frac{5}{16}$   $\frac{3}{8}$   $\frac{7}{16}$   $\frac{9}{16}$   $\frac{5}{8}$   $\frac{11}{16}$   $\frac{3}{4}$   $\frac{13}{16}$   $\frac{7}{8}$   
 $\frac{15}{16}$

Using 30 as a denominator:  $\frac{2}{15}$   $\frac{7}{30}$   $\frac{4}{15}$   $\frac{3}{10}$   $\frac{11}{30}$   $\frac{2}{5}$   $\frac{13}{30}$   $\frac{7}{15}$   $\frac{8}{15}$   
 $\frac{17}{30}$   $\frac{3}{5}$   $\frac{19}{30}$   $\frac{2}{3}$   $\frac{7}{10}$   $\frac{11}{15}$   $\frac{23}{30}$   $\frac{4}{5}$   $\frac{5}{6}$   $\frac{13}{15}$   $\frac{9}{10}$   $\frac{14}{15}$   $\frac{29}{30}$

## Generating the Lists

This is actually very easy. Choose a denominator, write all possible unsimplified aliquots, then work your way through the possible unsimplified sums, checking to see which are possible.

Example: for a denominator of 15, the possible aliquots are  $1/15$ ,  $3/15$  and  $5/15$ .  
So I could get  $4/15$ ,  $6/15$ ,  $8/15$  and  $9/15$ . The list, with fractions simplified where possible, would be  $4/15$ ,  $2/5$ ,  $8/15$  and  $3/5$ .

## Exploration 8: Solving Problems with Primes

### Directions for this Exploration

Your group will be given several different types of problems to solve. These are problems that are related to prime numbers in various ways as well as problems that are related to other topics in the explorations you have completed. Use any of your completed work from the explorations to help you in solving the problems. As you work together, first read each problem carefully, decide on the strategy you will use, then agree on how to implement the strategy. When you agree on the answer, write out your solution, including the strategy and all calculations. Be sure that your written solution is well organized and neatly done.

### Conclusion: Reflection on *Playing with Primes*

#### Directions for the Reflection

##### Part 1

Look back over your journal entries for the eight explorations. Make a list of five important mathematical ideas you learned during the explorations. Describe each idea in detail and list the places it was used in the unit.

##### Part 2

Write a paragraph describing your personal feelings about the unit. Which explorations did you like the most? Which did you like the least? Which were hard? Which were easy? Do you think the skills you learned here will be useful as you continue to study mathematics?

## Problems & Answers

### Primes

Find the least prime number greater than 2000. (2003)

*Bertrand's Postulate* states that there is at least one prime number between a counting number and its double. How many prime numbers are there between 25 and 50? (6)

What is the sum of the three distinct prime factors of 47,432? (20)

A *weird number* is a number that is the product of two consecutive prime numbers, such as  $7 \times 11 = 77$ . What is the least common multiple of the four smallest weird numbers? (Problem submitted by Matthew Mendicino) (2310)

A pair of *emirps* consists of two prime numbers such that reversing the digits of one number gives the other. How many pairs of two-digit *emirps* exist such that each number in the pair is greater than 11? (4)

For how many two-digit prime numbers is the sum of its digits 8? (3)

A positive integer has all single-digit prime factors. None of its prime factors occurs more than twice in its prime factorization, but the number is not a perfect square. What is the greatest possible number with these properties? (problem submitted by Todd Stohs) (22,050)

The numbers 3, 5 and 7 are important in many aspects of Japanese life. Most positive integers can be expressed as a sum of only 3's, 5's and 7's. For example, 15 can be expressed as  $5 + 5 + 5$  or  $3 + 5 + 7$ . What is the greatest even integer that cannot be expressed as a sum of 3's, 5's and 7's? (problem submitted by Diane Spresser) (4)

For how many prime numbers between 10 and 100 is the last digit also a prime number? (11)

How many positive three-digit prime numbers less than 200 have two digits the same? (7)

There are two prime numbers between 100 and 199 such that the tens digit is a prime number, the units digit is a prime number, and the tens and units digit taken

together are a two-digit prime number. Find the sum of these two prime numbers. (310)

The digits 1, 3, 4, 6, 8 and 9 are each used exactly once to form three two-digit primes. What is the sum of these prime numbers? (193)

What is the positive difference between the greatest and least prime factors of 2000? (3)

A three-digit number has the same hundreds, tens and units digit. The sum of the prime factors of the number is 47. What is the three-digit number? (777)

### Divisibility

For what value of  $n$  is the four-digit number  $712n$ , with units digit  $n$ , divisible by 18? (8)

For what value of  $n$  is the five-digit number  $7n,933$  divisible by 33? (5)

What is the least positive integer divisible by each of the first five composite numbers? (360)

For what digit  $n$  is the five-digit number  $3n85n$  divisible by 6? (4)

What is the least natural number that when divided by 11 gives a remainder of 0, but when divided by 2, 3, 4, 5 or 6 gives a remainder of 1? (121)

The six-digit number  $3730n5$ , with tens digit  $n$ , is divisible by 21. What is the value of the digit  $n$ ? (6)

For what single digit value of  $n$  is the number  $n5,3nn,672$  divisible by 11? (3)

The five-digit number  $4a,ab7$  is divisible by nine where  $a$  and  $b$  are single whole numbers. How many possible combinations are there for  $a$  and  $b$ ? (11)

## Fractions

From a bag of coins,  $\frac{1}{3}$  were given to Mary,  $\frac{1}{5}$  to Norm,  $\frac{1}{6}$  to Anna, and  $\frac{1}{4}$  to Bjorn. The six left were given to Troy. How many coins were originally in the bag? (120)

## Factors and Multiples

What is the greatest common factor of 2835 and 8960? (35)

What is the least possible positive integer with exactly five distinct positive factors? (16)

What is the least common multiple of 1537 and 1363? (72,239)

What is the product of the least common multiple and the greatest common factor of 22 and 48? (1056)

Find the Least Common Multiple of 12, 16 and 18. (144)

What is the smallest positive integer that has 8, 30 and 54 as factors? (1080)

## Miscellaneous

The number 35,335,872 is the product of several consecutive positive odd numbers. What is the greatest of these numbers? (17)

What is the sum of the fifth prime number, the sixth composite number, and the third perfect square? (32)

## Teacher Tips

Each group should be assigned the same set of problems so that solutions and strategies may be discussed. Remind the students that they have many resources to use – their individual work from this unit as well as the class charts. Reading through the problems as a class and discussing various strategies will help in giving the students the self confidence they need to tackle some of the more difficult problems.

# Ongoing Research Project

## Directions

Using any internet search engine, enter “Prime Numbers” in the search box. You will get hundreds, maybe even thousands of references. Explore some of these, and write down some key words to use in a narrower search. These words might be the names of mathematicians connected to primes or terms used when talking about numbers, like perfect or composite, etc. When you have written down at least five words, enter them, one at a time, into the search box. This should give you enough information on one topic to write a short report.

Since this will probably take more than one class session, be sure to write down the internet addresses of any sites you might want to visit again.

When you have enough information, write your report. It should contain any information relating to your topic that you found interesting. Be sure to read through it several times to check spelling and grammar.

At the end of your report, list the internet sites where you found your information.

## Bibliography

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<http://primes.utm.edu/glossary/index.php>
- Du Sautoy, Marcus. *The Music of the Primes*. Harper Collins Publishers, 2003.
- Friedberg, Richard. *An Adventurer's Guide to Number Theory*. Dover Publications, Inc., New York. 1994.
- Heinz, Harvey. *Patterns in Primes*. Copyright © 1998, 2000 by Harvey D. Heinz  
<http://www.geocities.com/~harveyh/primes>
- Mathcounts School Handbook*. Mathcounts Foundation, Alexandria VA. Various years.
- Weisstein, Eric W. "Prime Number." From *MathWorld*--A Wolfram Web Resource.  
<http://mathworld.wolfram.com/PrimeNumber.html>

## Internet Resources

- Prime Number Generator: <http://www.alumni.caltech.edu/~chamness/prime.html>
- Prime Strings Puzzle: [http://www.primepuzzles.net/puzzles/puzz\\_002.htm](http://www.primepuzzles.net/puzzles/puzz_002.htm)
- Eratosthenes History Page: <http://www.math.utah.edu/~alfeld/Eratosthenes.html>
- History of Prime Numbers:  
[http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Prime\\_numbers.html](http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Prime_numbers.html)
- Math History Web Sites: <http://homepages.bw.edu/~dcalvis/history.html>
- Divisibility Tests: <http://www.jimloy.com/number/divis.htm>
- Perfect Numbers:  
[http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Perfect\\_numbers.html](http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Perfect_numbers.html)

<http://home1.pacific.net.sg/~novelway/MEW2/lesson1.html>

List of the 10,000 smallest prime numbers:

<http://www.math.utah.edu/~alfeld/math/primelist.html>