

Proof as a Learning Tool in Algebra 1

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Overview

This document is primarily a research paper intended to inspire dialogue about the nature of mathematical proof and its role in a high school mathematics curriculum. What follows is a brief history of mathematics education and the place proof has held in that history from its introduction as a model of logical thought to its current almost non-existent role. I do not suggest that students are not being asked to think critically, communicate their reasoning, and justify their conclusions mathematically. My observation is, simply, that formal mathematical proof is not emphasized (in some cases, not taught) in most high school curricula and I argue that should not be the case.

Proof has traditionally been confined to instruction in Geometry classes. While proof of geometric theorems can reinforce logic and critical thinking, in college it is algebraic proofs that students are asked to construct. I propose that the introduction of basic algebraic proofs in the ninth grade Algebra 1 classroom would be beneficial to our students. In the classroom activities section I have included three lessons intended to serve as models for incorporating proof into the curriculum. The lessons are designed to follow one another in the order they are presented as they gradually ask the students to take on more responsibility for the structure of the proof. Also included is a mathematician biography activity which helps students learn about careers in mathematics, for which they will need an education that includes the ability to read and write mathematical proofs. The activities are intended to be taught in a collaborative, student centered classroom. Suggestions for journal entries are included, as I believe it is important for students to write in every class.

Rationale

Mathematical proof is at the core of the body of mathematical knowledge. An observable pattern is not a proof. Failure to find a counterexample despite

repeated attempts is not a mathematical proof. A proof is a logically sound argument that explains, using definitions and previously proven theorems why a given statement is true. Mathematical proof is a more rigorous requirement than scientific proof, which relies upon the ability to reproduce observable results.

Mathematics is sometimes perceived as an artless study of algorithms. Indeed, in the classroom, it has often been reduced to practice of mind-numbing number crunching, but the mathematical sciences are so much more than mere calculations. Over the last several thousand years, stretching back as far as we have written record, man has made observations about his environment and sought to find explanations for the phenomena he observes. Mathematical study has long been an aspect of this search for reason. The process of mathematics is such that an observation is made and a proposition is stated. The statement must then be proved mathematically, after which time it is not simply a proposition, but a theorem. For example, ancient Greek mathematician, Pythagoras observed that given any right triangle the length of the side opposite the right angle appeared always to be equal to the square root of the sum of the squares of the lengths of the other two sides of the triangle. He observed that this was true for every right triangle he had seen, but was not satisfied that this would be true for every right triangle that could possibly be constructed until he had created an indisputable proof. The proof made it clear that this relationship between the sides of the right triangle was true for *all* right triangles. For Pythagoras and his followers, there was evidence of divinity in the recurring mathematical truths they observed. In actuality, Pythagoras was not the first to make this observation, or to offer a proof of it, but it is his name that is given to the now famous Pythagorean Theorem.

Once a mathematical statement has been proved, it is called a theorem. If the proof is mathematically sound, the theorem can not be disproved without unraveling large pieces of mathematics. Theorems are proved by a logical sequence of statements, all of which are based on previously proven theorems. If the Pythagorean Theorem were decided to be a false statement, it would have to be because a mathematical relationship used to prove it was also deemed false and so on until we would have to begin again with the questions posed first by the earliest mathematicians. A proven theorem is an unalterable truth. Much of mathematical knowledge is applicable to the sciences, industry, investment and other arenas of life, but it need not be. The objective of mathematical study in its purest form is to discover and communicate mathematical truths. Proof is the language by which those truths are communicated.

Theorems and their proofs represent the building blocks of mathematical knowledge. The importance of proof in the study of mathematics cannot be overestimated, but where does it fit in to secondary mathematics education? Is it important that every student be able to read and write mathematical proofs or

should it only be required for college bound students or only for those interested in pursuing careers in science and mathematics? How rigorous should our requirements of proof be? In which classes should it be introduced and to what degree? These questions were first posed over one hundred years ago to a select group of educators and have been revisited periodically since then. The presence of mathematical proof in the secondary classroom has become less over time. I propose that we revive mathematical proof as a teaching and learning tool and that it be available to all students. What follows is a brief history of mathematics education and the pedagogy of teaching proof as well as my proposals for integrating proof into the Algebra 1 curriculum. I include two sample lessons for teachers to incorporate into their classroom that I hope will serve as a model for further lessons to be created by the teacher.

For most of the twentieth century our public schools have engaged in the practice of tracking students. Tracking, also called ability grouping, consists of classifying students according to the results of intelligence tests, grades and post-secondary goals. Students with aspirations and perceived qualifications to attend college upon graduating from high school are offered enrollment in upper-level college preparatory classes, while the remainder of students are offered “general” courses. Tracking was first established as an educational practice around the time of the industrial revolution. Prior to that time, only affluent families had the luxury of sending their sons to school. As an agricultural society, most families needed their children to work on the farms. Around the turn of the century our nation experienced a shift from agriculture to industry and with it an influx of immigrants. Urban areas grew virtually overnight and schools were built to serve them. The sons of affluent families were now joined by the children of immigrant factory workers, many of whom did not speak English. Tracking was devised as a way to meet the various needs of a diverse student body. Unfortunately, there was little distinction made at the time between limited English proficiency and limited intelligence. Immigrant children were channeled into lower tracks and were taught that their language and their culture were burdens they must overcome in order to be successful in America.

Another significant development in American education occurred in the late 1950’s. Russia launched Sputnik, the world’s first artificial satellite, on October 4, 1957. The response in the United States was largely disbelief that a nation we perceived as militarily powerful, but otherwise backwards and inferior had reached space before us. There was fear that the Russians would achieve a level of high tech military ability that would trump ours and that, unthinkable, the communists would “win” the Cold War. Congress responded by increasing federal funding for math and science programs in American public schools. It seemed that the communists were training more and better mathematicians and

scientists than we. It was at this time that the idea emerged that the citizens of a democracy need to be mathematically literate. Ironically, at this time many Black Americans and other minorities were still engaged in a struggle to have their right to a quality education recognized.

In the 1970's a movement grew among educators that criticized tracking as contributing to segregation and perpetuating the achievement gap between white and minority students. English teacher Jeannie Oakes, now vice chair of the Graduate School of Education at UCLA, argued that teachers of lower tracked classes have significantly lowered expectations of their students. Those teachers, she claims, are more likely to have students filling in blanks in worksheets instead of posing open ended problems that allow students to exercise creativity and critical thought. "Teachers in high-track classes are clearer, more enthusiastic, and use less strong criticism, and classroom learning tasks appear to be better organized and of greater variety." (Oakes, 1987, p. 142) It should be noted that tracking is also present in homogeneous schools and is no less harmful to the students designated to the lower tracks there.

The National Council of Teachers of Mathematics (NCTM) created an innovative document in 1989 entitled *Curriculum and Evaluations Standards for School Mathematics*. This document marked a historically important first step by a professional organization to articulate extensive goals for teachers and policy makers in a school discipline. NCTM echoed the sentiments of Jeannie Oakes and her peers; there will be no double standards in education. "These Standards describe an ambitious foundation of mathematical ideas and applications intended for all students. Through its emphasis on fundamental mathematical concepts and essential skills, this foundation would give all students solid preparation for work and citizenship, positive mathematical dispositions, and the conceptual basis for further study." (www.nctm.org)

This extensive document constituted a set of guidelines intended to facilitate mathematics education reform. It did not simply acknowledge that as educators we had let too many students graduate from high school without demonstrating mathematical proficiency. It did what educators and legislators had failed to do in the past by providing a clear explanation of what constitutes mathematical proficiency. Since 1989, the Standards, as they have come to be known, have been updated and revised several times. The most recent incarnation, published in 2001, outlined these expectations regarding reasoning and proof.

Reasoning and Proof Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

The response to the 1989 Standards was tremendous. Teachers and school administrators as well as parents and community members became informed advocates for educational reform. Among the reforms proposed was an effort to institute Algebra as a high school requirement for all students. The philosophy behind 'Algebra for All' is that students who take an Algebra course in high school are statistically more likely to enroll in and complete college. It is argued that Algebra also enables students to develop critical and abstract thinking skills. Data show that when Algebra is not a required course, minority students are less likely to enroll in an Algebra class than are their white peers. Mandating that all students take Algebra could lessen the achievement gap between white and minority students. Among the educators who have championed Algebra for all is civil rights activist and mathematics teacher, Bob Moses. Mr. Moses was instrumental in the 1960's campaign to organize black voters in Mississippi. In 1982 he founded The Algebra Project, an organization that works to help middle school students acquire the skills they will need to be successful in Algebra classes and beyond. "In today's world," Mr. Moses claims, "economic access and full citizenship depend crucially on math and science literacy." (Moses, 2001, p. 5)

Pittsburgh Public Schools is among the many school districts across the country that have mandated Algebra 1 as a graduation requirement. Since all students will take this course, it is an ideal course in which to introduce the fundamentals of mathematical proof. Traditionally, only about 50% of the students who voluntarily enroll in Algebra 1 receive a grade of C or better. As many school districts across the country began to mandate Algebra 1 for all students, this statistic has not improved, but has become evident to a greater number of parents. There are those who advocate for a return to less strict graduation requirements while others argue that it is the course that needs to be modified. Advocates of Algebra for all argue that better, more egalitarian texts are needed and that simply placing all students in traditional classrooms is not true reform. Our own district is currently debating about which curriculum will best meet the needs of all

students. I advocate the adoption of a non-traditional text such as Carnegie Learning.

Progressive texts distinguish themselves from more traditional curricula by containing opportunities for collaborative learning and investigations. They emphasize critical thought over application of algorithms and rarely require students to complete long pages of practice problems. There is no such thing as a perfect text that will meet the needs of all learners. However, it is my opinion, and one on which the activities contained in this document are dependent, that a collaborative, investigative approach to learning mathematics is more sound than a repetitive, algorithm oriented approach.

I hope that I have established at this point the self-evidence of the following three statements:

- Proof is fundamentally important to the construction of mathematical knowledge and to mathematics education.
- Algebra is, and should be, a required course for all students within the Pittsburgh Public School system (except those students with exceptionalities whose individual education plans determine otherwise).
- That Algebra course will best meet the needs of all students enrolled if a progressive text with an investigative, collaborative approach to mathematics learning is used.

The activities contained in this document were developed with the above principles in mind.

The role of proof in American secondary education has changed over time. In 1893, when the difficulties of educating a culturally and socio-economically diverse student body had only just begun to surface, a committee was organized to address the perceived need to create a standardized mathematics curriculum. The Committee of Ten was chaired by then Harvard president, Charles Eliot. The Committee's goal was to determine what the content of the high school curriculum should be. This decision was to be based largely on college admission standards. Mathematics curriculum recommendations were provided by The Mathematics Conference, an appointed board of six mathematics professors, two high school principals, and two high school math teachers. "Drawing on those recommendations, the Committee of Ten identified the high school geometry course as a vehicle for students to acquire *the art of demonstration*." (Herbst, 2002 p. 287) The recommendation was adopted by the Committee of Ten and thus, student proof construction became a required component of a Geometry class. This decision was significant because it required that students would construct proofs, rather than just read, memorize, and reproduce them. "The Committee argued that, as students had thus far been used to memorizing the

demonstrations of a geometry text, the mental discipline that geometry made possible was being lost.” (Herbst, 2002. p. 287)

Requiring students to construct proofs is not a simple task. Geometry textbooks at the turn of the 19th century contained proofs that were written in paragraph form. Students would study proofs of previously proven theorems and then be asked to construct a proof of a simple corollary to that theorem. It was assumed that the students would construct proofs in the style of the proofs they had studied. Educators found that students had great difficulty constructing proofs in this manner and over time textbooks were modified to facilitate proof writing by students. For example, some textbooks began to include diagrams of each step in the proof. The student’s job would be to provide the verbal justification of the illustrated step. Other textbooks made sure to limit the length of a proof to a single page, while still others moved away from writing proofs in paragraph form and instead used bullets to emphasize each distinct statement within the proof. In this way, textbooks gradually moved toward the presentation of all proofs in the two column format familiar to geometry students today. By requiring a very specific format educators believed that they were eliminating potential for error and misunderstanding. “The fact that the two-column format emphasized in so evident ways the formal aspects of proving, enforcing the notion that a proof consisted of steps of statements and reasons, made it useful at the time.” (Herbst, 2002, p. 298)

As a tradition of proving was being established in the geometry curriculum, mathematical rigor and formality grew to take precedence over creativity and original thought. Educator D.E. Smith determined that “whereas it was not reasonable to expect that all students would discover new truths, proving truths stated by somebody else was something that all students should be able to do.” (Smith, 1911, p. 160) The proof, rather than the proposition, became the issue of greatest importance. So, two-column proving solved the difficult pedagogical questions of how to teach proof and what to expect students to produce by way of a proof, but it did so at the expense of using proof as a knowledge construction tool. For many educators, this was too big a price to pay to meet the mandate that students must construct proofs in a Geometry class. NCTM’s Standards gave educators pause to reconsider the role that proof had assumed in the curriculum. Were their students ‘making and investigating mathematical conjectures?’ Were they ‘selecting and using various types of reasoning and proof?’ When these questions were seriously considered, many curriculums fell short.

In recent years, we have seen a shift in the role of proof in the Geometry classroom. The shift has been such that many students are never asked to construct formal proofs. This, in my mind, is no better than rote construction of proofs. I submitted a series of five questions about mathematical proof to 43

students at Peabody High School. The students were asked to answer the questions completely, honestly and anonymously. The conclusions that can be drawn from the results of this survey are obviously limited by the small sample size and the fact that all participants attend the same high school. Still, some conclusions can be drawn from the results. See Appendix for copy of the survey.

The participating students were in the following grades and classes.

Ninth Grade - 4 students, Honors Geometry

Tenth Grade - 7 students, Geometry

1 student, Algebra 1

1 student, Algebra 2

Eleventh Grade - 1 student, Geometry

24 students, Algebra 2

Twelfth Grade - 5 students, Algebra 2

In response to the first question, “What is Mathematical Proof?” twenty-six students responded that they did not know or responded with answers that made no sense at all. (i.e. a stack of pancakes) This left only 17 out of 44 respondents (about one third) who attempted to answer the question. Eleven of those remaining stated that mathematical proof is the process of checking your work. These students appeared to understand that to prove is to confirm or verify something, to determine that it is true, but they clearly had not been introduced to formal mathematical proof in the classroom. Two of the remaining six respondents indicated that proof had something to do with showing that triangles are congruent. This is one of the only contexts in which proof is consistently used in Geometry classrooms today. Significantly, the only students responding with answers indicating a sophisticated understanding of mathematical proof were the four ninth grade respondents enrolled in an honors Geometry course. Each of these four students indicated that a proof uses “formulas” and given statements to show that a mathematical statement is true. The responses to this first question indicate that formal proof is emphasized in the honors curriculum, but not in the mainstream geometry courses.

Other questions posed by the survey asked whether or not the students construct mathematical proofs in the class they are currently taking or in classes they have previously taken and how often. The lack of knowledge of what a mathematical proof is, as indicated by the responses to the first question, rendered most responses to the subsequent questions useless. So, what can be concluded by the responses to these surveys is that most of these students are not asked to construct formal proofs, two-column or otherwise. Are they then, not being asked to explain their reasoning in their mathematics classes or to justify the procedures they choose to solve posed problems? I do not believe this to be the case. I believe that mathematical reasoning is alive and well, but formal proof has been

all but abandoned by mainstream high school Geometry curricula.

Proof, when it is taught, is used to reinforce theorems that are usually irrelevant to the students. Students are rarely given an opportunity to engage in the mathematical process of observing a pattern, making a prediction, justifying their prediction mathematically, and submitting their justification for review by their peers. “Interactions are a vital component in providing students with experiences through which they restructure their knowledge of mathematics.” (Pugalee, 2001, p. 2) Educational research indicates that most students learn best through collaboration and investigation. We know now that only about 10% of people learn by simply listening to verbal instruction, yet many teachers cling to the lecture technique in the classroom. Of particular inspiration to me when designing these activities was an article written by Geometry teacher, Rhonda Cox, published in the January 2004 issue of NCTM’s journal, *Mathematics Teacher*. In this article, Mrs. Cox explains how she developed a geometry unit in which students constructed quadrilaterals, made observations based on measurements or angles, sides and diagonals, formulated conjectures and wrote proofs of these conjectures. This unit was completed cooperatively with little intervention by the teacher. Because the students were proving their own conjectures they were invested in the outcome. Mrs. Cox witnessed not only an improvement in students’ ability to construct logical arguments, but also in their disposition toward the subject. It is on this model that I have based the development of materials contained in this document.

Objectives

These activities were developed for use in a ninth grade Algebra 1 classroom to accomplish the following goals:

- Students will learn what mathematicians do and how the body of mathematical knowledge is developed.
- Students will develop greater understanding of Algebra 1 mathematical concepts. This will be due, in part, to increased feelings of ownership of and investment in mathematical ideas because they will be formulating conjectures based on their own observations.
- Students will be able to construct logical mathematical proofs in a variety of formats.

Students often perceive mathematics as a tool to be applied to other areas of studies. The question, ‘when are we going to use this?’ is familiar to all math

teachers. The applications of mathematics are extremely important and do help students contextually understand the skills they are acquiring, but too often we dismiss mathematics as a valuable area of study in its own right. Most high school students have vague ideas of what electrical engineers or architects or physicists do and yet have no concept of how a mathematician spends her days. One goal of these activities is to acquaint students with the process of doing mathematics for its own sake.

NCTM's reasoning and proof standards are contained earlier in the rationale of this document, but they are not the only standards intended to be met by these activities. As these activities are intended to supplement an Algebra 1 course, the Algebra standards for grades 9-12 are intended to be met as are the communication standards (see Appendix). It is my opinion that no assessment can demonstrate mathematical understanding more completely than student communication. The second communication standard asks that all students be able to "communicate their mathematical thinking coherently and clearly to peers, teachers, and others." (NCTM, 2001) Of these, communication between peers is perhaps the most valuable. As students develop conjectures and defend them to their peers, they will be internalizing and retaining the mathematical concepts far better than they would by simply performing calculations using these concepts. Furthermore, by working to prove their own conjectures, they will have a greater investment in the outcome.

Finally, students will be able to construct logical mathematical proofs in a variety of formats. They are welcome to use the two-column proving method, but are not restricted to it. They will be encouraged also to include diagrams and pictures to illustrate mathematical properties. The proofs students construct in undergraduate and graduate mathematics courses bear no more resemblance to two column proofs than a skyscraper to a tree house. By introducing a more comprehensive version of proof, students will be better prepared to engage in post-secondary study of mathematics.

Strategies

Entering ninth grade students have conceptions of proof based on their experiences outside of school. They understand what it means to prove one's point, but they have not yet been asked to construct logical proofs of conjectures in a mathematics class. They have been asked to prove or disprove a hypothesis in a science class in the middle grades, and it is this concept of proof that we must develop further. When introducing activities in which students will read and construct mathematical proofs, it is first necessary to agree upon a student generated definition of mathematical proof. The teacher should help students understand the role that proof plays in mathematics in general and if possible,

should invite a mathematician to share the details of his or her job with the class. Students should see that in the process of constructing mathematical knowledge, proof is the last step. The process begins with making observations. This step should be done collaboratively. In small groups, students should study a list of operations, or several graphs or equations. Whatever they are studying, the teacher's job is simply to provide the materials. The students must formulate conjectures based on the materials. The key to implementing these activities is that the students must work collaboratively and that they must be the creative force behind the activities. They must formulate the conjectures and they must prepare the arguments to defend their conjectures. The teacher's role is that of facilitator, not lecturer.

Working in small groups, students will formalize their arguments and present them to the class. Their peers will have opportunity to study the prepared proofs and to ask questions or raise objections to the material. The group presenting then has opportunity to respond and make any necessary revisions. As the students' theorems are proved, they will be added to a posted list of class theorems. Through this process of discovery, the mathematics will be made more meaningful to the students and will prevent the common misperception that mathematics is a series of complex rules that are beyond the comprehension of average people.

Algebraic proof often amounts to manipulating algebraic expressions and determining that one expression is equivalent to another expression. Some students will be able to perform these manipulations without making errors and may still not understand the relationship they are demonstrating, while other students may be able to explain a mathematical relationship verbally in a way that demonstrates real understanding, but may lack the arithmetic skills to express it symbolically. "A proof, as valid as it might be in terms of formal derivation, actually becomes both convincing and legitimate to a mathematician only when it leads to real mathematical understanding." (Hanna, 2000, p. 7) It is up to the teacher to determine whether or not the "proof" provided by the student leads to real mathematical understanding. I expect that this determination will vary between students.

Classroom Activities

1. Who are Mathematicians and What Do They Do?

Objectives:

- *Students will begin thinking about the job of a mathematician as well as the study of mathematics.*
- *Students will be able to identify at least one living mathematician and the contributions they have made to their mathematical field*

•*Students will develop an understanding of mathematics as something that exists beyond the high school curricula.*

Class Time: 1 day researching (Internet access required)
1 day poster preparation (should be finished at home, if necessary)
1 day presentation and reflection

This is a research project and can be incorporated at any time throughout the school year. If introduced during Women's History Month or Black History Month it provides opportunity for raising awareness of diversity issues in the academic world. It should be completed before beginning formal proving activities as it helps students see a human element to mathematics. Also, the activity is fun and may dispel some student anxieties.

Introduction: The question, 'Who are mathematicians and what do they do?' should be posed as a question to the class before beginning the project. Students may respond to it in the form of a journal entry or as a 'think, pair, share' activity in which the students 'think' about the question individually, 'pair' up with a partner and discuss their thoughts before 'sharing' with the rest of the class.

Research: Students work in pairs to complete the **My Mathematician** research worksheet. It is helpful to provide students with a suggested list of Internet resources as well as some books. A brief list of resources is included at the end of the lesson.

Presentation: Following the research process, the partners will prepare an informative poster with information about the mathematician's life and career. To the extent possible, the student should explain the area of mathematics in which the mathematician works.

Students should then be given an opportunity to look over their peers' posters. This is vital as it sets a precedent of peer review that will be expanded upon throughout the unit. The teacher should provide them with the **Mathematician Scavenger Hunt** worksheet to complete. The Scavenger Hunt worksheet provided is only an example. The teacher should create questions based on the posters prepared by the students in the class.

Reflection: Finally, the question initially posed as an introduction should be asked again. This time, students should submit a written response to the teacher.

Suggested Student Resources - Books and Websites:

Henrion, Claudia. *Women in Mathematics: The Addition of Difference*

Indianapolis: University of Indiana Press, 1997

Perl, Teri. *Math Equals: Biographies of Women Mathematicians + Related Activities*, Menlo Park, CA: Addison Wesley Publishing Corporation, 1978

Mathematicians of the African Diaspora - <http://www.math.buffalo.edu/mad/>

Careers That Count: Opportunities in the Mathematical Sciences - <http://www.awm-math.org/ctcbrochure/toc.html>

Association for Women in Mathematics - <http://www.awm-math.org/>

American Mathematical Society - <http://e-math.ams.org/>

Note to teacher If you wish to complete a model project, I highly recommend selecting Andrew Wiles, the Princeton mathematician who solved Fermat's Last Theorem. Allow a class period to watch The Proof (Nova documentary) and lead a class discussion about what a proof is. The students will probably be surprised to learn that Dr. Wiles spent over seven years writing a single proof.

MY MATHEMATICIAN

Mathematician's Name

Date of Birth _____ Place of Birth _____

Education: College or University

Degree Earned

Current Job: _____

Area of Mathematics: (Briefly describe the area of study and its applications)

Interesting Biographical Information:

EXTRA CREDIT: email your mathematician! Tell them who you are and why you are contacting them. Ask them questions about their job and the mathematics they study. Send a copy of your email to your teacher.

MATHEMATICIAN SCAVENGER HUNT

Using the posters created by your classmates, find a mathematician who...

was born before 1950 (name and date of birth) _____

was born after 1950 (name and date of birth) _____

was born in the United States _____

was born in an African nation _____

is a woman _____

lives in Pennsylvania _____

wrote at least two books _____

has the same birthday as you or someone in your family _____

fought in a war _____

has more than five children _____

does math that applies to medicine _____

does math that has military applications _____

speaks more than one language _____

plays a musical instrument _____

Who is your favorite living mathematician that you have learned about? Why?

2. Add It Up

Objectives: •*Students will be introduced to the process of making an observation, stating a proposition, and constructing a proof.*
 •*Students will be introduced to Gauss Theorem and learn the proof of it.*

Class Time: This activity can be completed in one class period. If using the Carnegie Learning Curriculum it can supplement or replace the Gauss lesson which is placed early in the curriculum, following the students' introduction to symbolic notation and n th terms.

Introduction: The teacher should provide students with some biographical information about Gauss. This information continues to add a human element to the mathematics. Gauss is a very interesting character who could perform complex mathematical tasks at an early age.

Student Work: Students will complete the **Add It Up** worksheet individually. This should take only a few minutes. It is likely that none of the students will determine the formula for the sum of n integers. The class should discuss their observations and speculations before beginning the **Add It Up Twice** worksheet in small groups or pairs.

After students have had a reasonable amount of time to work on **Add It Up Twice** the class should again regroup and students should present their solutions and observations. The teacher's job is to facilitate discussion and maintain order, but not to provide solutions or confirmation of the correctness of solutions. It is important to facilitate the development of a mathematical community among the students. They should offer feedback to each other. **The Proof** is the final worksheet to be completed by the students for this lesson. As this is their first construction of a mathematical proof, students will require teacher assistance.

Reflection: The teacher should emphasize that writing proofs is a three step process involving observation, proposition, and proof. The students *observed*, in this instance, that when adding the string of integers from 1 to a set number, the sum is equal to the product of the largest integer and one more than that integer divided by 2. The class then *proposed* that observation would be true for any string of integers 1 to N and they created a symbolic representation of that observation by writing $N * (N+1) \div 2$ (or an equivalent expression). Finally, they constructed an algebraic *proof* of their proposition.

The teacher may wish to have the students keep a list of class theorems that have been successfully proved. This list could be in the students' journals or posted on the wall.

Subsequent lessons: Adding the first n integers is a good introduction to sequences and series for an Elementary Functions or Calculus class. Constructing a proof of the rule to find the sum of the first n squares ($1 + 4 + 9 + \dots + n^2$) requires knowledge of the inductive method of proving, which I do not recommend for ninth grade students, but would be an excellent exercise for a Calculus course.

ADD IT UP

Carl Friedrich Gauss was a German mathematician born in 1777. Even as a small child he had astounding mathematical ability. When he was eight years old his teacher gave the class the following assignment.

Add up all of the integers between 1 and 100.

Without the use of a calculator Gauss responded immediately that the sum is 5050 -- His stunned teacher checked the sum and found he was correct! How did he do it?

Add all of the integers between 1 and 2

$$1+2 =$$

Add all the integers between 1 and 4

$$1+2+3+4=$$

Add all of the integers between 1 and 10

$$1+2+3+4+5+6+7+8+9+10=$$

Do you see a pattern yet? Is there a shortcut for adding up all of the integers from 1 to n or did Gauss use magic?

ADD IT UP TWICE

Gauss had the idea that he should add up the string of numbers TWICE and then divide the sum by two. Here is how he set it up.

Let's add the integers from 1 to 5

$$\begin{array}{r}
 1 + 2 + 3 + 4 + 5 \\
 + \quad 5 + 4 + 3 + 2 + 1 \\
 \hline
 \quad + \quad + \quad + \quad + \quad =
 \end{array}$$

Add Vertically

All of the vertical sums are the same value. What is that value? _____

How many are there? _____

What is TWICE the sum of the integers 1 to 5? _____

This total value is TWICE the sum of the integers from 1 to 5, so Gauss knew to divide that value by two. What is the actual sum of the integers from 1 to 5? _____

Try 1 to 10

$$\begin{array}{r}
 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 \\
 \hline
 \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad =
 \end{array}$$

Add Vertically

What is the value of the vertical sums? _____

How many are there? _____

What is TWICE the sum of the integers from 1 to 10? _____

What is the sum of the integers from 1 to 10? _____

Let's try 1 to 20 without writing out every step.

$$\begin{array}{r}
 1 + 2 + \dots + 20 \\
 + 20 + 19 + \dots + 1 \\
 \hline
 \end{array}$$

What do each of the vertical sums add up to? _____

How many are there? _____

What is twice the sum of the integers from 1 to 20? _____

What is the sum of the integers from 1 to 20? _____

Do you see a pattern yet? Is there a shortcut for adding up all of the integers from 1 to n ? If so, what is it?

THE PROOF

Observation - Explain the pattern you observed for adding up the integers from 1 to n.

Proposition - State the shortcut as a formula you believe will work for ANY value of n.

Proof - Prepare the formal Mathematical Proof

$$\begin{array}{r} 1 + 2 + \dots + N \\ + N + N-1 + \dots + 1 \\ \hline \end{array}$$

What will each of the vertical sums add up to be? _____

How many will there be? _____

What is TWICE the sum of the integers from 1 to N? _____

What is the sum of the integers from 1 to N? _____

Explain the steps of your proof.

3. Deductive Reasoning

Objectives: *•Students will begin thinking about what Mathematical Proof is*
•Students will define deductive reasoning
•Students will review rules of integer multiplication and addition
•Students will prove that the product of a positive and a negative is a negative

Class Time: This activity should take one class period

Introduction: The question, “What does it mean to prove something?” is posed to the students as a ‘think, pair, share’ activity. Student responses are shared and recorded on the chalk board. The teacher should ask students to extend their concept of proof to mathematics. The students should generate all of the ideas and suggested definitions of proof. However, the teacher should make sure that students understand that repeated examples are not a proof.

Instruction: The teacher will introduce the idea of a ‘given.’ That is, a statement that we already know to be true. The teacher will then help students understand how given statements can be used to make logical conclusions by posing examples such as the following:

Given: My home is three miles from the school.
The grocery store is between the school and my home.

Conclusion: Therefore the grocery store is less than three miles from the school/my home.

The teacher should introduce the vocabulary term DEDUCTIVE REASONING at this time. Deductive Reasoning is the process of using facts (givens) to lead logically to a new fact (conclusion). It should be emphasized that deduction is not the only way to prove a conclusion. It is one method that may be employed.

Student Work: The students are asked to perform several integer multiplication problems and to observe that the product of a positive and a negative appears to always be a negative value. When the class has agreed upon this observation, they are informed of their given statements. Each statement should be thoroughly discussed and examples should be done to insure that students understand them. Students will record their given statements on the **Prove It!** worksheet and will construct the proofs in pairs.

Given: Additive Inverse Identity $a + -a = 0$

Given: Distributive Property $a(b + c) = ab + bc$

Reflection: Students will write a brief journal entry responding to the question, “What is deductive reasoning and how did you use it in class today?”

The students should keep a list of class theorems either in their journals or posted on the wall.

****Note to teacher**** The proofs contained in the answer key are sophisticated, but with some guidance, I believe that our students will be able to understand and generate these proofs. Alternatively, if you want students to *justify*, but not really prove, that the product of a positive and a negative number will be a negative number, consider inductive reasoning such as the following:

$$3(4) = 12$$

$$3(3) = 9$$

$$3(2) = 6$$

$$3(1) = 3$$

$$3(0) = 0$$

$$3(-1) = ?$$

The students should be able to observe the pattern of the products decreasing by three as the multiplier decreases by one. They should also be able to logically extend the pattern to negative multipliers. This approach will relate multiplication to linearity and will reinforce perception of multiplication as repeat addition.

PROVE IT!

Find the following products.

$3(-3) =$

$3(-2) =$

$3(-1) =$

$-3 + 3 =$

$-2 + 2 =$

$-1 + 1 =$

$3(-3 + 3) =$

$3(-2 + 2) =$

$3(-1 + 1) =$

Observations:

Deductive Proof:

Given Statements:

1. _____

2. _____

Use the given statements to prove that $3(-2) = -6$

Use the given statements to prove that $6(-5) = -30$

Use the given statements to prove that $a(-b) = -(ab)$

PROVE IT! (Answer Key)

Find the following products.

$$3(-3) = -9$$

$$3(-2) = -6$$

$$3(-1) = -3$$

$$-3 + 3 = 0$$

$$-2 + 2 = 0$$

$$-1 + 1 = 0$$

$$3(-3 + 3) = 0$$

$$3(-2 + 2) = 0$$

$$3(-1 + 1) = 0$$

Observations: Students should observe that all of the products are negative. They may also observe that the multiples of three increase by three and the multiples of six decrease by six. This is a good observation to follow up on later if you wish for students to provide an inductive justification for this integer multiplication rule.

Deductive Proof

Given Statements: 1. Additive Inverse Identity.

2. Distributive Property

for each of the given statements, students should write an explanation or give examples.

Use the given statements to prove that $3(-2) = -6$

Consider $3(-2 + 2)$. We know that this is equal to $3(-2) + 3(2)$, by the distributive property. We also know that it is equal to $3(0)$ (which equals 0), by the additive inverse identity. So, $3(-2) + 3(2) = 0$. $3(2) = 6$, so $3(-2)$ must equal -6 .

Use the given statements to prove that $6(-5) = -30$

See above proof, but modify for given values.

Use the given statements to prove that $a(-b) = -(ab)$ where a and b are positive numbers

Consider $a(-b+b)$. This equals $a(0) = 0$ (by Additive Inverse Identity) and it also equals $a(-b) + ab$ by the distributive property. So, $a(-b) + ab = 0$. Solving for $a(-b)$, we see that $a(-b) = -ab$

4. One line, many equations

Objectives:

- *Students will review evaluating algebraic expressions and solving algebraic equations*
- *Students will manipulate algebraic expressions to create equivalent algebraic expressions*
- *Students will understand that equivalent equations can be applied to the same linear model*

Class Time: One class period.

Warm Up: Given a graph of a line, students will identify the coordinates of points on the line and determine the slope of the line. Students perform this task individually and relate their methods and results to the class. This activity will reinforce the fact that given a line, the slope between any two points on that line is constant.

Instruction: Using the linear graph provided for the warm up, the class should select one point and write the equation of the line using the point-slope format. $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point on the line.

The students will then individually pick other points on the line and write linear equations in point-slope form. The students should compare their equations and discuss whether or not they are all equations of the line shown in the graph. Ask the students to justify their proposition that these equations are all equivalent.

If no student has chosen the point $(0, b)$ (the y -intercept), ask them to do so. They will have generated the slope-intercept equation of the line in this way. Is this also an equivalent equation? How can they tell?

Student Work: In groups the students should construct a proof that $y - y_1 = m(x - x_1)$ is equivalent to $y = mx + b$ for any line.

They should identify known information as given statements and construct a logical progression of ideas.

Possible Solution:

Given - We know that the slope between any two points on a line (x_1, y_1) and (x_2, y_2) is given by the ratio $(y_2 - y_1)/(x_2 - x_1)$. We use m to represent the slope.

Given - we know that if we can identify the slope and one point on a line, we can express that line as $y = m(x - x_1) + y_1$

Proposition - If we know the slope of a line and the y -intercept that line we can represent the line using the equation $y = mx + b$, where $(0, b)$ is the coordinates of the y -intercept

Proof - we can choose ANY point on the line to write the point-slope equation of the line, so let's choose the y -intercept $(0, b)$. Then, using point-slope form the equation of the line is $y = m(x - 0) + b$ which is the same as $y = mx + b$

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Appendix- Content Standards

Contents

1. Survey
2. NCTM Reasoning and Proof Standards
3. NCTM Algebra Standards
4. NCTM Communication Standards

1. Survey

Please answer the following questions completely, honestly and anonymously. I intend to use your responses in a research paper I am writing on the role of mathematical proof in high school classes. I appreciate your cooperation very much.

Thank you,
Ms. Press

Name _____ Grade _____

Math class you are taking now _____

1. What is Mathematical Proof?
2. Do you use mathematical proof in the class you are taking now? If so, how often?
3. Did you use mathematical proof in previous math classes? If so, which ones?
4. Do you like writing proofs? Why or why not?
5. Use this space to include any comments that were not covered by the previous questions.

2. NCTM Reasoning and Proof Standards (www.nctm.org -- all citations refer to bibliographical information contained on this site)

Reasoning and Proof Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to --

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

3. *NCTM Algebra Standards (www.nctm.org)*

Algebra Standard for Grades 9-12

UNDERSTAND PATTERNS, RELATIONS AND FUNCTIONS

- generalize patterns using explicitly defined and recursively defined functions;
- understand relations and functions and select, convert flexibly among, and use various representations for them;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions;
- understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions;
- interpret representations of functions of two variables

REPRESENT AND ANALYZE MATHEMATICAL SITUATIONS AND STRUCTURES USING ALGEBRAIC SYMBOLS

- understand the meaning of equivalent forms of expressions, equations, inequalities, and relations;

- write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency-mentally or with paper and pencil in simple cases and using technology in all cases;
- use symbolic algebra to represent and explain mathematical relationships;
- use a variety of symbolic representations, including recursive and parametric equations, for functions and relations;
- judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.

USE MATHEMATICAL MODELS TO REPRESENT AND UNDERSTAND QUANTITATIVE RELATIONSHIPS

- identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;
- use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
- draw reasonable conclusions about a situation being modeled.

ANALYZE CHANGE IN VARIOUS CONTEXTS

- approximate and interpret rates of change from graphical and numerical data.

4. *NCTM Communication Standards (www.nctm.org -- all citations refer to bibliographical information contained on this site)*

Communication Standard for Grades 9-12

Instructional programs from prekindergarten through grade 12 should enable all students to --

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas clearly