Investigating Fractals

Lesley Britton

Taylor Allderdice High School

Contents of Curriculum Unit

Overview

This document contains objectives, guidelines for lectures, and materials needed to instruct advanced high school mathematics students in the basic structure of fractals, the algorithms involved in creating fractal pictures, fractal dimension, and leading them in a research project where they will investigate the basic properties and algorithms involved with fractals.

Objectives

Students will be able to define and discuss the basic properties of a fractal. Students will investigate one specific type of fractal in depth. They will research the algorithm for producing the fractal, the dimension, and the properties of the fractal. After researching, students will create an artifact that shows their fractal for at least 10 generations and create a presentation discussing what their research revealed about their fractal.

Fractal topics will include:

- Pythagorean Tree
- Spirals
- Koch's Curve
- Sierpinski's Triangle
- Sierpinski's Carpet
- Dragon Fractal

- Cantor’s Dust
- Sierpinski’s Sponge
- The Mandelbrot Set
- Newton Iteration
- Fern leaf

Rationale

Although fractals are not a large part of the math curriculum in the city of Pittsburgh, this lesson is nice for advanced placement (AP) mathematics or science students at the end of the year, after the test has been administered. It is always a challenge keeping students focused and interested in these last few weeks after the test they have worked so hard to study for is finally over. Most students are ready to be “done” and are not very highly motivated at this point in the year. This lesson provides something new and interesting enough to keep my students challenged in these last few weeks. Since the idea of fractals has crept more into the general public, my students have recently become more aware of them and are interested in how they relate to the math they already know. Benoit Mandelbrot states in his book <u>Fractals, Graphics, and Mathematics Education</u> that fractals “enchant many young people and make them excited about learning mathematics and physics.” I have found this to be true. Students at any level can easily have their interest sparked from just looking at a few pictures. This quickly leads to their wanting to know the mathematics involved.

In any group of students there are going to be diverse interests. Even in an advanced mathematics or science class not all students will be planning to pursue a career in that field. However, all students can benefit from it. Mandelbrot points out that the uniqueness and excitement of fractals “can help make these subjects easier to teach to teenagers… even those students who do not feel they will need mathematics and physics in their professions.” Just as those who are interested in mathematics should broaden their interests to art, music, and the like, those interested in the liberal arts might be inclined to take an interest in a mathematical topic. By teaching fractals to a group of students with varying interests, I hope that it will help to create citizens that see mathematics as challenging yet achievable and pass this belief on. Presenting a topic that is different from the mainstream mathematics students are used to can easily interest people who would normally shy away from higher mathematics.

At the beginning of the unit, the students should be expected to have little or no previous knowledge of the topic of fractals. This unit consists of several different stages where they will be given information through lecture, research to find information on their own, and finally present what they have found.

Since students know very little about the topic, the first part of the unit is teacher directed through lecture and discussion. I will give the class a basic knowledge of what a fractal is and is not, how a fractal is formed and also a brief discussion of fractal dimension. This should take one to two days of classroom lecture. All of this information is included in the paragraphs to follow.

Once the students have an understanding of what a fractal is they will begin the independent research component of the project. Here, they will choose a specific fractal that they are interested in and research how the fractal is formed, its dimension, and generate several generations of the fractal in a visual aid that will be used for the presentation part of the project.

When research is complete, each student will prepare a classroom presentation where they will show their classmates exactly how their fractal is generated, present their finished product and discuss its dimension. The presentation portion of the project will not only motivate them while they are researching, but also allow them to share their knowledge with groups of people who share their interest and excitement of this topic.

For my advanced placement students, I feel that this method of learning through mostly independent research is best for several reasons. During the last few weeks of school these students, mostly seniors, are very busy with wrapping up loose ends and preparing for graduation. Allowing them to use class time to begin research and then finish on their own time makes this project a lot more flexible for both teacher and student. More importantly, however, I have found that these particular students are willing and able to use their resources to find more information on their topic of choice than I could ever present to them. By allowing each of them to choose the fractal they want to investigate, I allow them to find something they are interested in, which is an enormous motivator. When I do this project with my students, they are genuinely excited about their fractal and what they are finding. They begin asking questions that they would never have thought of. This interest leads to groups of students discussing, researching, and comparing what they know and have found. They become interested in each others research and it is amazing to see how much effort they put forth when they have been given the freedom to choose a topic they are interested in.

Especially with my advanced placement students, this method of teaching also becomes a learning experience for me. When my students begin their projects I am their main source of information. However, after a few hours on the computer, they begin to find fractals I have never seen and ask questions that I can then research with them. It is a great learning experience for teacher and student and it allows an opportunity for teacher and student to truly work together as a team.

When my students present their projects after research is complete, I am amazed at how much they are interested in each other's presentations. They are constantly comparing their fractal to what is being presented, starting discussions, and learning from one another. For these students, I believe that they learn the most when given the freedom to investigate what they are interested in within the topic.

Background Information

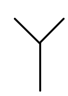
Fractals are geometric patterns that are repeated over and over again at smaller and smaller scales to produce irregular shapes and surfaces. What makes them different is that they cannot be described or produced using classical geometry. Instead, a fractal pattern requires recursive use of a basic generator (or basic building block of the fractal) to build it.

A generator is a shape made from more basic pieces that can be formed in classical (Euclidean) geometry. A generator can be as simple as the "Y" shape created when a tree branches. To form a fractal using this generator, we simply start with a straight line and then branch off using the "Y" shape. Then each new end branches off using the "Y" shape again and so on. The more times the branching is done, the more complicated the figure becomes. The figures below show this branching effect using the "Y". (Nuhfer)

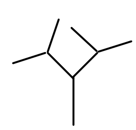
Figure 1: The "Y" Generator

Initial Line

After using the "Y" generator one time



After using the "Y" generator two times



Self-Similarity

Fractals all have a property called self-similarity (also called self-symmetry). This means that any tiny piece of the fractal looks just like the original piece. You can see this in the "Y" branching above. If you zoom in on a tiny piece of the complex form, you can still see that same "Y" branching that you see in the first generation. Again looking to nature for an example, we can see this self-similarity very clearly in the growth pattern of a fern. If we take a look at the branches and leaves of a fern we see that each tiny branch that sprouts from a main branch looks exactly like the bigger branch that it is a part of. It is simply a much smaller version.

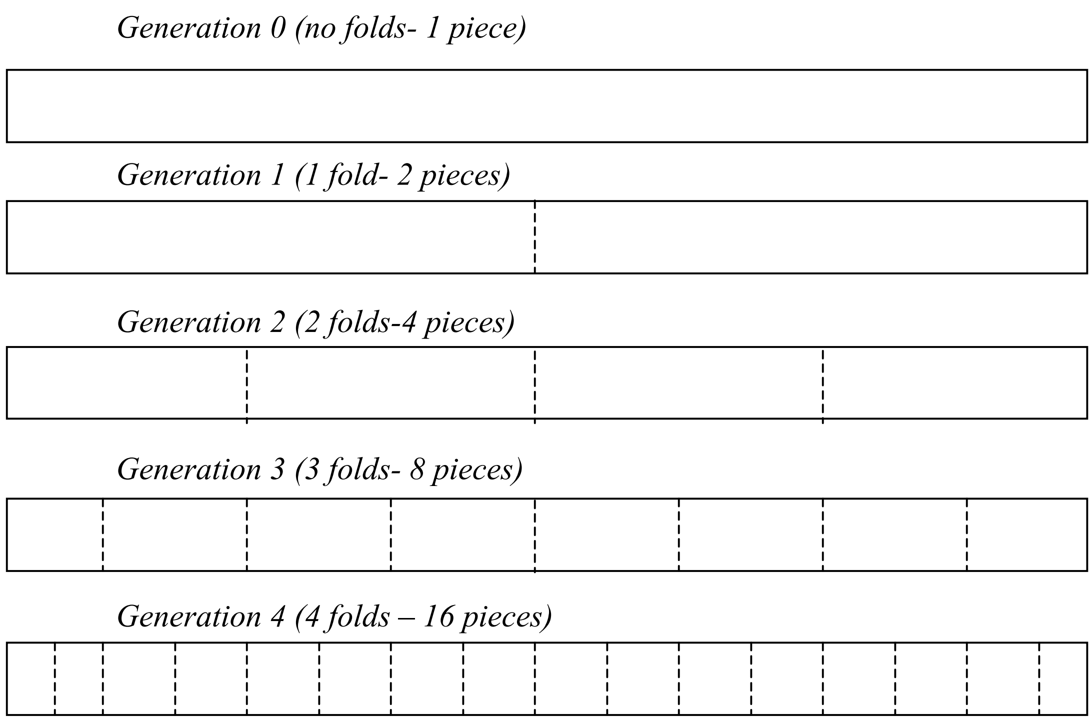
Figure 2: Image of a Fractal Fern.
http://motivate.maths.org/conferences/conf1/c1_intro_to_fractals.shtml

As with the fern, as a fractal picture is created, the same process is done over and over again. Usually, this makes a very interesting, and often very attractive, design. For each fractal the algorithm is different, but the repeating self similarity is always the case.

Creating a Fractal

Each fractal has a different algorithm (set of rules) for creating it. In the "Y" fractal above the rule was simply to branch every new line into a "Y" shape. The Dragon fractal is also useful as an example of a fairly simple algorithm. The Dragon fractal is made by dividing a line of finite length in half, then dividing the two new smaller pieces in half, dividing those four even smaller pieces in half, and so on. The result is that you get shorter and shorter segments as you make more and more folds. It is very easy to see the fractal nature of this because at any time in the process you can look at one piece and it is “similar” to the original line. It’s just a tinier version of the original. This can easily be done as a class activity by giving each student a long strip of paper and having them fold it in half over and over.

Figure 3: The first four generations of the Dragon Fractal. Dotted lines represent folds.



Though some fractals have more complicated algorithms than this, they all have their own unique set of rules that is used repeatedly to create the fractal image.

Fractal Dimension

In classical geometry it is easy to identify the dimension of an object (things can have no dimension like a point or they can be one, two, or three dimensional). Fractals, however, can have non-integer dimensions. While this may seem odd at first, we must remember that the dimension of an object means much more than whether something has length, width, height or all three. Though we will not go into the details of dimension here, it is necessary to note that non-integer values are possible. In very simple terms, we can think of dimension as a measure of how much "stuff" is there and not necessarily just the shape of things. (Deustch)

One thing we will always know about the dimension of a fractal is that it cannot be greater than the medium in which it is created. For example, if it is possible to draw a fractal on a two-dimensional sheet of paper, then the fractal must have a dimension less than or equal to two.

We will develop the equation for calculating the exact fractal dimension by looking at the first two generations of a few very simple fractals: a line, a square, and a cube. Though very common and simple, these all are considered fractals because it is possible by repeating the same algorithms over and over again to create smaller pieces that look exactly like the original.

We will start with the line. The one-dimensional line can be broken into two smaller similar pieces simply by cutting it in half. Therefore, in the second generation we have two pieces that have a scale factor of two (if you double their length they would be back to the size of the original).

The two dimensional square is very similar. It can be broken into four smaller similar pieces by making two cuts through the center; one horizontal and one vertical. In this case, the second generation has four pieces that have a scale factor of two (if you double all of the side lengths they would be back to the original's size).

The three dimensional cube can be broken into eight smaller similar pieces by making three cuts through the center; one horizontal and two perpendicular verticals. Now the second generation has eight pieces that, again, have a scale factor of 2 (if you double all of the side lengths of each small cube they would be back to the size of the original).

	Scale Factor (f)	Dimension (d)	Number of Pieces (n)
Line	2	1	2
Square	2	2	4
Cube	2	3	8

It is clear from the table above that the dimension is simply related to the number of pieces by the equation

$$f^d = n$$

Though this equation is handy, what we really want to do is to be able to find out what the dimension of a specific fractal is. In other words, we need to solve this equation for d. Since d is in the exponent, we will need logarithms to do so.

$$f^d = n$$
$$\log f^d = \log n$$
$$d \log f = \log n$$
$$d = \frac{\log n}{\log f}$$

This equation allows us to calculate the dimension of a fractal simply by knowing the number of pieces we have and the scale factor.

To test this equation, we can look back at the Dragon fractal. We originally have a line. After one generation, we have two smaller lines. So we have two pieces with a scale factor of two.

$$d = \frac{\log 2}{\log 2} = 1$$

The Dragon fractal is one-dimensional. This is not a surprise since the Dragon fractal is simply a straight line.

Using this basic knowledge of creating and investigating fractals, students will have enough of a knowledge base to begin their research. They will first choose which fractal they would like to investigate. The guide sheet in Appendix A should be given to guide them in their research. They should begin by finding the algorithm for creating their fractal and then filling in the table. This table will be different for each fractal. Completed tables for several fractals are included in Appendix B for teacher reference.

Strategies and Classroom Activities

As most students have little or no knowledge of fractals, the first days of this lesson will be teacher directed. Class lecture and discussions will include the definition of a fractal, the general algorithm for creating a fractal picture, the idea of a fractal dimension, and the procedure for calculating the dimension of a fractal. It is important the students understand the idea of having fractal dimension that is not an integer value. It is a good idea to allow students some time to become comfortable with these ideas. Derive the equations (derivation shown above) for fractal dimension and have them use it to calculate the dimensions of fractal they know (like the line, square, and cube).

When students have chosen their topics, they will need several days to begin researching their topic. I use class time for this, but if your schedule does not allow it, they can certainly research independently. After a few days, it is a good idea to regroup in the classroom and have each student discuss his or her progress. Some will have stumbled upon information on other people's topics and this can be shared at this time. This should be an opportunity for students to present what they have so far as well as questions and difficulties that can be addressed to the class.

Upon completion, each student's research will result in the creation of the first five generations of their fractal accompanied by a seven to ten minute presentation on the production, nature, properties, and dimension of their fractal. The guide sheet in Appendix A should be used as a basis for this presentation.

Appendix A: Student Research Guide

Fractal Topic: ____

1. Describe how your fractal is formed. (Outline the algorithm).

Fill in the table below.

Generation	Number of Pieces	Length/Area of Each Piece	Total Length/Area
0	1	1	1
1			
2			
3			
4			
N			
∞			

2. What about the nature of this makes it a fractal?

3. What is this fractal's dimension? (Show your calculation and explain your work.)

4. List and discuss the properties of this fractal?

Appendix B: Examples of Filled out Guide Sheet

Fractal Topic: ____Dragon Fractal____

1. Describe how your fractal is formed. (Outline the algorithm).

   The Dragon Fractal is formed by folding a line in half over and over again.

Fill in the table below.

Generation	Number of Pieces	Length/Area of Each Piece	Total Length/Area
0	1	1	1
1	2	½	1
2	4	¼	1
3	8	1/8	1
4	16	1/16	1
N	2^n	$1/2^n$	1
∞	∞	0	1

2. What about the nature of this makes it a fractal?

   In each generation, the pieces get smaller and more numerous. However, the smaller pieces are always similar to the original (self-similarity). The pieces are created by a repeating the same algorithm. Eventually, at infinity, there are an infinite number of pieces of 0 length.

3. What is this fractal's dimension? (Show your calculation and explain your work.)

After 1 generation, we have 2 pieces that have a scale factor of 2 (doubling their size would bring them back to the original again.) Therefore, $n = 2$ and $f = 2 \ldots$

$$d = \frac{\log n}{\log f} = \frac{\log 2}{\log 2} = 1$$

Since the dragon fractal is clearly a one-dimensional line this makes perfect sense.

4. List and discuss the properties of this fractal.

The Dragon fractal is one dimensional, has an infinite number of pieces that have zero length and has a total length of one. (The final length is exactly the same as the length you started with.)

Fractal Topic: Koch's Curve

1. Describe how your fractal is formed. (Outline the algorithm).

   Koch's Curve is constructed by dividing a line segment into three congruent pieces and then replacing the middle piece by two sides of an equilateral triangle (the sides of the triangle must be the same length as the removed piece of the line). This is repeated on each new line (there are 4 now) of the curve.

Fill in the table below.

Generation	Number of Pieces	Length/Area of Each Piece	Total Length/Area
0	1	1	1
1	4	$\frac{1}{3}$	$\frac{4}{3} = 1.\overline{3}$
2	16	$\frac{1}{9}$	$\frac{16}{9} = 1.\overline{7}$
3	64	$\frac{1}{27}$	$\frac{64}{27} = 2.\overline{370}$
4	256	$\frac{1}{81}$	$\frac{256}{81} = 3.16046$
N	4^N	$\frac{1}{3^N}$	$\left(\frac{4}{3}\right)^N$
∞	∞	0	∞

2. What about the nature of this makes it a fractal?

   Koch's curve is created by repeating the same process over and over again. In the end you have an infinite number of pieces of zero length. The dimension of Koch's curve is fractional.

3. What is this fractal's dimension? (Show your calculation and explain your work.)

After one generation, there are four pieces that have a scale factor of 3 (tripling their size would bring them back to the size of the line before them). Therefore $n = 4$ and $f = 3\ldots$

$$d = \frac{\log n}{\log f} = \frac{\log 4}{\log 3} = 1.261859507$$

Since Koch's Curve is drawn on a 2 dimensional page, we know its dimension must be less than 2.

4. List and discuss the properties of this fractal?

Koch's Curve has an infinite number of pieces that have zero length. Its total length is infinity. It has a dimension of about 1.26.

Appendix C: **Student Assignment**

FRACTAL PROJECT

FOR THIS PROJECT YOU WILL BE REQUIRED TO:

1. CHOOSE A FRACTAL THAT YOU WOULD LIKE TO RESEARCH.
2. COMPLETE THE STUDENT RESEARCH GUIDE FOR THE FRACTAL THAT YOU CHOOSE.
3. PREPARE A 3-5 MINUTE CLASS PRESENTATION THAT DISCUSSES THE ALGORITHM FOR CREATING THE FRACTAL, THE PROPERTIES, AND THE DIMENSION OF THE FRACTAL.
4. DRAW OR OTHERWISE CREATE A MINIMUM OF 4 GENERATIONS OF YOUR FRACTAL (NOT INCLUDING THE 0 GENERATION) TO BE USED AS A VISUAL AID IN YOUR PRESENTATION.

- YOU WILL HAVE 2 CLASS PERIODS TO RESEARCH IN THE LIBRARY ON _______________ AND _______________.

- YOUR COMPLETED STUDENT RESEARCH GUIDE AND VISUAL AID ARE DUE ON ___________________.

- YOU WILL PRESENT ON ____________________________.

Fractal Project Rubric

Name ______________________

Research Guide Sheet

Outline of algorithm	___________ / 2 points
Table Completed and Correct	___________ / 10 points
Description of fractal nature	___________ / 2 points
Fractal dimension calculation	___________ / 4 points
Properties	___________ / 4 points

Artifact

First 4 generations	___________ / 10 points

Presentation

Discussion of algorithm	___________ / 2 points
Discussion of dimension	___________ / 2 points
Discussion of properties	___________ / 2 points
Discussion of artifact	___________ / 2 points

_________ / 40 points

Works Cited

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