# Patterns in Nature: The Logarithmic Spiral and Parabolic Trajectories

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## Overview

This unit is intended to be used in an Elementary Functions class, with 11th and 12th graders. The unit has two parts. In the first part, students will identify and quantify the mathematical relationship found in a spiraling whelk shell. In the second part, students will identify and quantify the mathematical relationship found in each trajectory of a bouncing golf ball, and then model the decreasing maximum height of each bounce using exponential decay as a function of time and as a function of the bounce number. They will determine the percent lost per second, and the percent lost per bounce, and then consider which of these two models is more appropriate.

## Rationale

In Elementary Functions, students study linear and quadratic functions, exponential functions (both exponential growth and decay), logarithmic functions, and trigonometric functions. They study both Cartesian and Polar coordinate systems. They also learn a technique called re-expression of data to identify various types of functional relationships. Each of these topics or areas is studied in relative isolation. The patterns found in a spiraling whelk shell, and in the trajectories of a bouncing golf ball, provide a context where the students must consider all of these topics simultaneously. The purpose of these experiments is to uncover an application rich in relevance, where the various topics interact, and the students are forced to draw upon each topic where appropriate. Students will re-express the data gathered by measuring the growing radius of the seashell to identify the type of correlation, and then algebraically manipulate the linear equation to find the original function. This technique is introduced in our textbook Precalculus, Graphical, Numerical, Algebraic, by Demana, Waits, Foley, and Kennedy, in Section 3.4, "Properties of Logarithmic Functions". The example which the textbook provides calls for students to verify Kepler's Third Law of Planetary Motion, which states that the square of the period of the orbit T for each planet is proportional to the cube of its average distance D from the sun. This law is beautiful in its simplicity, but I wanted the students to re-express data which they had gathered themselves. These two experiments provide just that opportunity.

These applications are also rich in rigor. The layering of the tasks reinforces the deeper connections between the various areas of math mentioned above. Finally, the use of the graphing calculator translates into a technological proficiency which will be essential for these students next year in Calculus.

## Objectives

Whelk Experiment:

- Use a ruler to measure the radius of each rotation of a whelk shell.
- Enter this data into a pair of lists in the graphing calculator.
- Re-express this data using natural logarithms to identify which re-expression generates a linear function.
- Perform a linear regression calculation on the lists which are in a linear correlation.
- Translate this linear equation back into a function correlating the radius with the number of rotations.
- Repeat these steps to find a function correlating the radius with the angle of rotation (expressed in radians).
- Translate this equation in Cartesian coordinates into polar coordinates.
- Graph the polar equation over a Scatter plot to verify the correlation.

Golf Ball Experiment:

- Use the Calculator Based Ranger (CBR) to generate a Scatter plot of the position of the golf ball during several bounces.
- Use the vertex and one passing through point to find a function for each trajectory.
- Create a list of each maximum height, the number of the bounce, and the time that height occurred.
- Use exponential regression to correlate the maximum height to the bounce number.
- Use exponential regression to correlate the maximum height to the time.
- Identify the percent lost per bounce, and the percent lost per second.
- Defend which of the two models is more appropriate.

## Strategies

Students will work in pairs during each of the experiments. I will pair a stronger student with a weaker one, so that peer to peer instruction will occur. I will facilitate as little as possible, to allow the students to work through the lab on their own. I will only re-direct or assist when absolutely necessary and I will do this as little as possible. My goal is for the students to uncover the hidden relationships without my guidance. Each pair will present their findings to the class at the conclusion of the experiment.

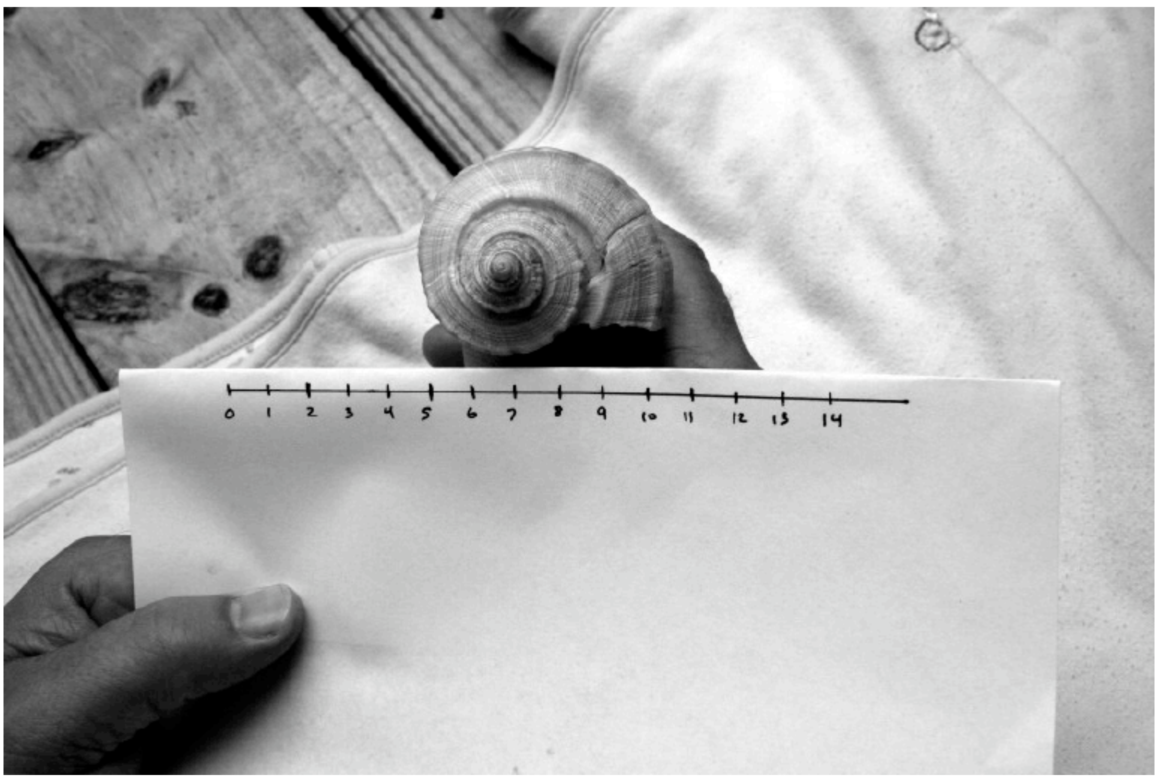
**<u>Classroom Activities</u>**

These experiments will be performed after Section 3.4 in the textbook is completed. Students will thus have some experience re-expressing data using natural logarithms. (Important: Directions to students will be italicized.)

**WHELK SPIRAL**:

PART I: CARTESIAN COORDINATES

Students will work in teams of two. Each team will receive a photograph of a whelk shell, taken from above.

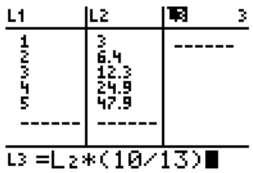



0 1 2 3 4 5 6 7 8 9 10 11 12 13 14


3 4 5 6 7 8 9 10 11 12 13

*Draw a pair of axis on your photograph, with the origin at the center of the spiral. Record the radius (in mm) for each of the first five rotations along one of the axis. Use the ruler in the photograph to scale your measurements of the photograph to the actual radii of the shell.*

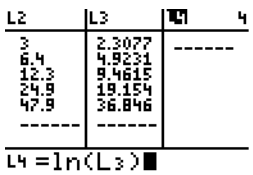
List 2 contains the measurements of the photograph, while List 3 contains the radii of the actual shell because the 10 cm ruler on the photograph measures 13 cm, so the scale factor is 10/13. This transformation of the data will not be obvious to the students, so there will be a class discussion at this point in the experiment as to how to scale the photograph measurements to the radii of the actual shell.



| L1 | L2 | L3 |
|---|---|---|
| 1 | 3 | ------ |
| 2 | 6.4 | |
| 3 | 12.3 | |
| 4 | 24.9 | |
| 5 | 47.9 | |
| ------ | ------ | |

L3 =L2*(10/13)

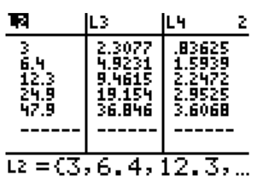
List 3 contains the radius of each of the first five rotations, in mm, of the actual shell.



| L2 | L3 | L4 |
|---|---|---|
| 3 | 2.3077 | ------ |
| 6.4 | 4.9231 | |
| 12.3 | 9.4615 | |
| 24.9 | 19.154 | |
| 47.9 | 36.846 | |
| ------ | ------ | |

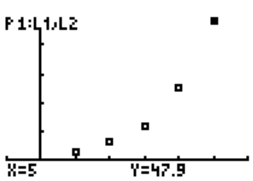
L4 =ln(L3)

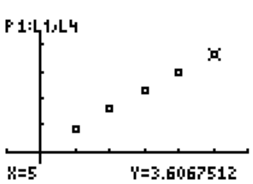
*Re-express the data using natural logarithm until you find a linear correlation.*



| L2 | L3 | L4 |
|---|---|---|
| 3 | 2.3077 | .83625 |
| 6.4 | 4.9231 | 1.5939 |
| 12.3 | 9.4615 | 2.2472 |
| 24.9 | 19.154 | 2.9525 |
| 47.9 | 36.846 | 3.6068 |
| ------ | ------ | ------ |

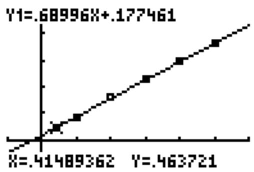
L2 ={3,6.4,12.3,...



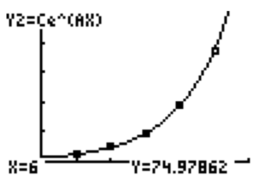



The graph of List 1 vs. List 4 is linear. This means that there is an exponential relationship between List 1 and List 3.

*Use linear regression to determine the equation of the line.*




*Use this equation to find a function for List 1 vs. List 3.*




The equation, which correlates the rotation number to the radius of the shell, is shown here, where C is 1.19418 and A is 0.68996. Students should find this equation by solving:

$$\ln(Y) = AX + B.$$

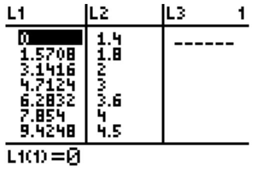
$$e^{(\ln(Y))} = e^{(AX + B)}$$

$$Y = e^{(B)} * e^{(AX)}$$

$$Y = C * e^{(AX)}$$

The scatter plot above shows that the curve fits the data very well.

*Record the radius for every quarter rotation. Enter the rotation amount (in radians) in List 1, and the radius in List 2.*

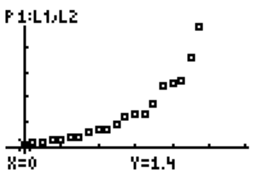


| L1 | L2 | L3 1 |
|---|---|---|
| 0 | 1.4 | ------ |
| 1.5708 | 1.8 | |
| 3.1416 | 2 | |
| 4.7124 | 3 | |
| 6.2832 | 3.6 | |
| 7.854 | 4 | |
| 9.4248 | 4.5 | |

L1(1)=0

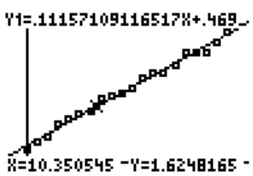
List 1 shows quarter rotations (in radians), while List 2 shows the radii.

*Draw a scatter plot of List 1 vs. List 2.*

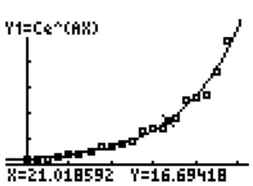



There seems to be a periodic deviation in the data, due to the spiral of the shell turning off-center. I would look for that observation during the student class discussion.

*Re-express the data using natural logarithm until you find a linear correlation.*
*Use linear regression to determine the equation of the line.*




*Use this equation to find a function for List 1 vs. List 3.*




In the exponential equation, A = .11157 and C = 1.598. The graph shows that the curve fits the data very well.

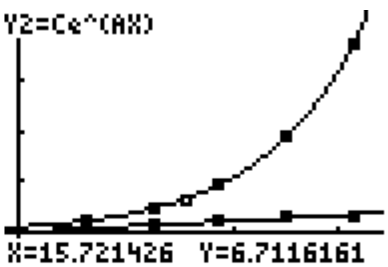
PART II: POLAR COORDINATES

*Enter the radian equivalent for 1 rotation, 2 rotations, etc. in List 1. Enter the radius for these rotations in List 2. Make List 3 the natural logarithm of List 2.*

| L1 | L2 | L3 1 |
|---|---|---|
| 6.2832 | 2.3077 | .83625 |
| 12.566 | 4.9231 | 1.5939 |
| 18.85 | 9.4615 | 2.2472 |
| 25.133 | 19.154 | 2.9525 |
| 31.416 | 36.846 | 3.6067 |
| ------ | ------ | ------ |

L1(1)=6.283185307…

*List 1 vs. List 3 is linear. Find the equation of this correlation. Use this equation to find the correlation between List 1 and List 2.*

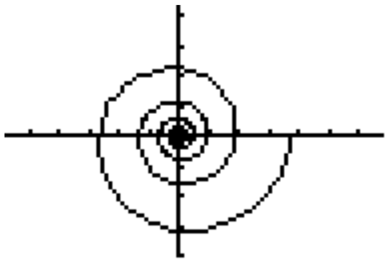



The linear equation is y = .10981x + .17747

The exponential equation, correlating List 1 and List 2, is y = Ce^(AX), where C = 1.19419 and A = .10981

*You have correlated the radius to the amount of rotation (in radians), where y is the radius and x is the amount of rotation. In polar mode, graph r = Ce^(AΘ).*

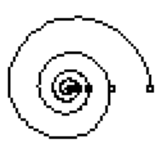
```
WINDOW
 θmin=0
 θmax=31.4159
 θstep=.1
 Xmin=-57.89483…
 Xmax=67.894836…
 Xscl=10
↓Ymin=-40
```

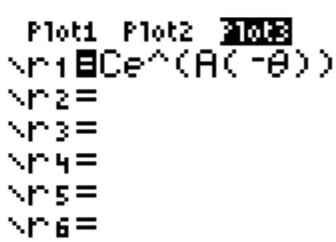


*Create List 4, a list of 5 zeros.  Turn off the axis, and graph the spiral again, with a scatter plot of List 2 vs. List 6.  This will show the 5 radii along the x-axis.*

This graphic shows that the correlation is very strong.  The curve “hits” the 5 radii very closely.

*Did you notice that the shell is spiraling in the wrong direction?  Can you adjust the equation and window to make the shell spiral in a clockwise fashion?*





```
WINDOW
 θmin=0
 θmax=-31.4159
 θstep=-.1
 Xmin=-5
 Xmax=50
 Xscl=10
↓Ymin=-35
```

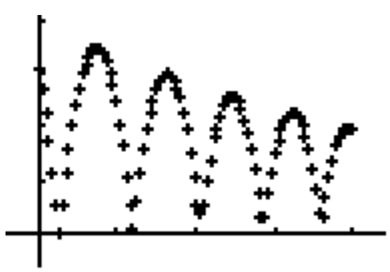
This change in direction is difficult to accomplish. To do it, the student must change theta to negative theta in the equation, and then change the window to reverse the direction of the theta increments. I hadn't figured it out myself when I tried the experiment out on my CAS Elementary Functions Class. One student suggested the above solution. I told him I didn't think it would work. I said, tongue in cheek, "go ahead and try. Fall flat on your face in front of all of your friends. That's fine by me." He tried it and it worked. The class erupted in cheers when it worked, I was totally embarrassed, and he became something of a folk-hero in the school as word spread of his success. The excitement generated by this experiment exceeded my expectations.

In conclusion, I note that indeed the polar exponential equation models the shell quite well.

**BALL BOUNCE**:

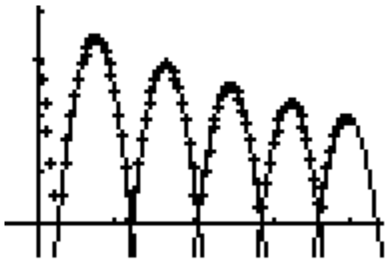
New teams will be formed, with two students per team. Students will go into the hallway, and use the CBR to capture data of at least 5 bounces of a golf ball. When all teams have captured the data, they will return to the classroom.

*Graph a scatter plot of List 1 vs. List 2.*



*Identify the vertex of each parabola, and enter the coordinates in List 3 and List 4. Find an equation for each parabolic trajectory using the vertex and one other data point on each curve.*

Students will algebraically determine the five parabolas.



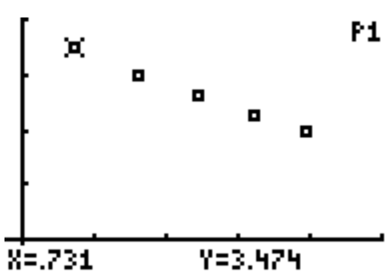
The five equations for my particular trajectories were:

$$Y = -15.947(X - .731)^2 + 3.474$$
$$Y = -16.042(X - 1.634)^2 + 2.977$$
$$Y = -15.761(X - 2.451)^2 + 2.593$$
$$Y = -15.725(X - 3.225)^2 + 2.263$$
$$Y = -14.978(X - 3.956)^2 + 1.994$$

*Average the five values of the vertical stretch factor to find the acceleration of gravity according to your data.*

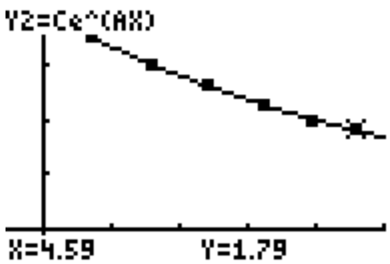
In my case, the five values average to -15.691, making the acceleration of gravity -31.38 feet per second squared. This value is pulled down by the fifth equation, which was lower than expected. If I throw out the fifth value, and average only the first four, I get -31.74 feet per second squared, which is very close to the actual value of 32 feet per second squared. Students will discuss possible sources of error in gathering the data, most likely of which is not keeping the CBR device at a constant height during the experiment.

*Graph List 3 vs. List 4.*



*This curve represents exponential decay. Re-express the data into a linear format, and use linear regression to find the equation of the line. Then use that equation to find the exponential equation for List 3 vs. List 4.*


This graph of List 3 vs. ln (List 4) shows a strong linear correlation. Therefore, List 3 vs. List 4 is exponentially correlated.




This graph shows the exponential decay of the maximum height vs. time. The equation above has C = 3.943, and A = -.172.

*Rewrite the exponential function as a function to the "x" power to find the percent of the maximum height lost per second.*

Y = 3.943* e^(-.172X)
Y = 3.943* ((e^-.172)^x)
Y = 3.943* (.842)^x

This form of the equation represents a loss of 15.8% of the maximum height per second.

This models the data very well.

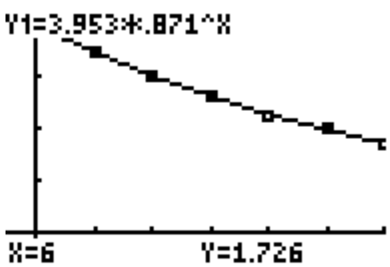
*Determine the percent of maximum height lost per bounce.*

Students must correlate the decay to the number of bounces, so they must create a List 6, representing the bounce number.

| L4 | L5 | L6 6 |
|---|---|---|
| 3.474 | 1.245 | 1.000 |
| 2.977 | 1.091 | 2.000 |
| 2.593 | .953 | 3.000 |
| 2.263 | .817 | 4.000 |
| 1.994 | .690 | 5.000 |
| ------ | ------ | ------ |

L6 ={1.000,2.000…

List 6 vs. List 4 will show an exponential correlation.




This shows that the ball loses 12.9% of its maximum height per bounce.

Students will then debate which model is more appropriate for the data, a percent loss per second or a percent loss per bounce. I would argue that much more energy is lost in the act of bouncing than to air resistance during flight, so a loss per bounce would be more appropriate. But I would not advance that opinion myself, and see if the students propose that argument themselves.

At this point, I would make the teams perform the experiment again in the hallway using a less efficient ball (perhaps a tennis ball), with a higher percent lost per second and per bounce. They would perform the calculations again using the new data, and determine the percent lost per bounce with the new ball.

Resources:

Dehaene, Stanislas. The Number Sense: How the Mind Creates Mathematics. 1997, Oxford University Press

Hamming, R.W. The Unreasonable Effectiveness of Mathematics. February 1980, The American Mathematical Monthly, Volume 87

Huntley, H.E. The Divine Proportion: A Study in Mathematical Beauty. 1970, Dover Publishers, NY

Maor, Eli. 'E', The Story of a Number. 1998, Princeton University Press

http://www.notam02.no/~oyvindha/loga.html

http://www-groups.dcs.st-and.ac.uk/%7Ehistory/Curves/Equiangular.html

http://en.wikipedia.org/wiki/Logarithmic_spiral

http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Erbas/KURSATgeometrypro/golden%20spiral/logspiral-history.html

http://online.redwoods.cc.ca.us/instruct/darnold/CalcProj/Fall98/DarrenT/EquiangularSpiral.html