

## **Problem Solving Through Probability in Higher Mathematics**

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**Overview**

Learning about the counting principle, permutations, combinations, and probability can prove to be very difficult for the pre-calculus mathematics student. Typically, students can understand and apply the necessary formula when prompted. However, students struggle when faced with an application that is more difficult in nature. This unit will emphasize a problem solving approach to probability, through small group explorations and discussions, that will help develop a deeper appreciation and understanding for this particular branch of mathematics. The many real life applications of probability will serve as motivation for students, and ensure that their level of interest in the subject remains at its highest level.

This unit will also include ten “Problem of the Month” problems, to be offered for extra credit in the elementary functions course. The problems will be challenging in nature, appropriate to this level of mathematics, and displayed on a bulletin board for an entire month of school. These problems will challenge students much in the same way that as problems done in class on a daily basis. Students will hopefully see the connection between problem solving skills and mathematics success.

**Rationale**

I had forgotten why I loved mathematics so much, and what drove me to be successful mathematically until I attended my first Pittsburgh Teachers Institute seminar on Problem Solving: Where Education Interacts with Life. We did several challenging problems, and I felt like a kid in a candy store. It was like I had rediscovered my love for mathematics and problem solving for the first time.

I was excited, as were the rest of the fellows, to be able to come up with the solutions to these problem solving activities. I wondered if doing these same types of activities in my class could generate the same type of excitement in my students. It is my hope by developing the problem solving skills of my students that they will feel more comfortable and confident in trying the more challenging problems that we do as part of our Elementary Functions curriculum.

We must understand that problem solving is the cornerstone of any successful mathematics program, and then try to infuse an enthusiastic feeling and attitude about it into our daily teaching (Posamentier, Smith, & Stepelman, 2006). The Pittsburgh Teachers Institute seminar on problems solving has helped me realize this in a short period of time. During the seminar, I would take some of the problems that we were given in class back to my classroom, and let my students work on some of the problems. Some problems were very simple in nature, but still offered a challenge to my students. I offered extra credit for the problems, thus serving as an extra motivator. My students immediately took a liking to these types of problems, and actually request them on a weekly basis. They enjoyed the investigations and multiple solutions of the problems. I immediately saw the level of interest in mathematics grow exponentially in a short period of time. It is my goal to replicate this as much as possible throughout the school year.

I have been teaching the Elementary Functions course at Schenley High School for three years. Elementary Functions serves as a pre-calculus course for our students. This class consists of students who are in eleventh and twelfth grade and have successfully completed Algebra II with a grade of "C" or better. The students in this class are all college bound students, mostly in eleventh grade. The next math course for students in Elementary Functions is calculus, whether it is taken in college (for those students in twelfth grade) or high school (for those students in eleventh grade).

Currently, I teach the Pittsburgh Scholars Program (PSP) sections of this course. These are some of the most mathematically talented that we have at Schenley High School. They are not, however, the kind of students that are driven by pure mathematics. In other words, most of them are good at math, but a lot of them do not care for the subject. During my first three years of teaching this course, I was sick of hearing the question "When are we ever going to use this stuff?" I tried to explain to my students that doing mathematics was a lot like solving a puzzle, similar to doing a brain teaser. I often used this explanation and justification because that is what drove me to be successful in mathematics. I liked solving puzzles and figuring out things that most people could not. I enjoyed challenging problems, and problems that related to real-world data. It was as if those problems were more meaningful because they had some

connection to the real world. The carryover into life outside of school adds importance to the mathematics our students study, and will ultimately improve their everyday performance (Posamentier, Smith, & Stepelman, 2006).

Current research further reiterates the point that problem solving is essential in the mathematics classroom. The National Council of Teachers of Mathematics, NCTM (2000) indicates that “problem solving is not only a goal of learning mathematics, but a major means for doing so.” As educators, we would like all of our students to be adept at problem solving. Realizing that students can learn through problem solving is a strategy that is not often used in most of our current curriculums. Problem solving is not only a skill to be taught and used in mathematics, but it is also one that will be carried over to everyday decision-making situations (Posamentier, Smith, & Stepelman, 2006). We do introduce techniques of problem solving early in our current elementary functions curriculum, and they are used throughout the year.

Currently, problem solving is taught in our current textbook (*Precalculus: Graphical, Numerical, Algebraic*) in the first section of the first chapter, using George Polya’s four problem solving steps. The four steps are understand the problem, devise a plan, carry out the plan, and look back (Demana, et.al., 2001). The four steps are used in solving applications throughout the textbook. Each time that a problem is solved, the text lists the steps related to the problem solving process: model, solve, support, and interpret. In this unit, I want to take the idea of problem solving to a new level, using it as a basis of instruction.

The next part of the process was trying to figure out where I could squeeze some problem solving activities into an already rigorous curriculum. This year, our PSP classes were extended to sixty minutes in length. I had primarily been using the time to review homework solutions and do extra examples. There are a number of problem solving activities that can be done in a short period of time, so that I can still teach my lesson for that particular day without sacrificing the pace of the course. However, I really wanted to try and find a few areas where the problem solving activities were related to a topic that we were learning. I found a number of resources on probability, and I thought that it would be the perfect fit for my current curriculum.

Typically, our students struggle with probability and statistics school-wide. It is always an area of weakness that we must review before the PSSA (Pennsylvania System of School Assessment). Maybe part of the problem is that our kids never apply what they learn, or use it in any useful way. My plan is to re-develop the unit on probability and statistics that I currently teach, including some problem solving activities and applications that I can find to relate to what we learn in class. I believe that this will be a great benefit for the students. Not

only will it challenge them and help them improve their critical thinking skills, but it will also be tied to an important topic that is part of our curriculum, leading to a deeper understanding of the topic of Probability. I firmly believe that this will generate excitement for some of the students who may find my class boring. I want them to see that mathematics can be fun, even when it is hard.

My plan for this unit is to change the way I currently teach chapter nine of our textbook: Discrete Mathematics. Typically, I introduce the counting principal through one example, followed by several additional examples to reinforce the concept. Then, combinations and permutation formulas are given to students, with rationale for using each formula at the appropriate time. Next, students are assigned several problems to complete. However, students do not seem to grasp and apply these concepts to my desired level of performance. Further exploration is needed to teach these concepts in a proper way, and this unit will ensure that all students understand the concepts at a much higher level.

The topic of probability is developed in a similar manor. Students at this level have been exposed to some basic probability, so the topic is not new to them. I start by defining the sample space and the event, followed by the definition of probability and several examples. The problem lies in students calculating the sample space (the set of all possible outcomes of an experiment). To do this, students need to rely on their knowledge from the previous lesson, which has not been developed fully. It is my hope that a new approach, one based on problem solving and further exploration with real world examples, will result in students coming out of this unit with a higher level of understanding.

The unit will be taught in the second nine weeks of the school year, after the completion of chapter two, before the end of the semester. The time frame for this unit is approximately two weeks. I will use the current textbook (*Precalculus: Graphical, Numerical, Algebraic*), along with some supplementary materials to develop the topics in a new way. Almost every topic that is covered in the textbook will be covered with the supplementary material, except for the concept of conditional probability, which I will teach using the current textbook. The level of problems with the supplementary material and the current text are quite similar, so I do not believe that I will be losing anything by taking this alternative route. Students will be given minimal instruction during the lesson, but instead will be working in pairs or small groups on a series of explorations. At the end of each section we will summarize what we have learned, and students will be assigned homework problems from the textbook. The student explorations will vary in difficulty, easy enough for the lower-achieving students, but difficult enough to challenge the most gifted student.

Another key component of this unit is the fact that the new lessons will be student-centered as opposed to teacher-directed. This will be a huge shift in lesson design, compared to how the class is currently taught. Our school district is pushing the student-centered lesson, and these explorations are just one more way for me to implement this strategy into my classroom. Too often, I find myself in front of the classroom for 60 minutes lecturing, as you would see in most college-level mathematics classes, or other classes of this level. As teachers, we sometimes think that the mathematics is too difficult for students to discover, and discovery lessons are difficult to implement. In these supplementary exercises, I will have the students work in pairs or small groups to complete the problems, and then we will meet in the last 20 minutes of class to discuss the results. I will ask probing questions, engaging students in accountable talk and academic rigor. We will listen to each other, hear what everyone in the class has to say, and build off of other students' connections.

To further generate interest in problem solving and challenge my students, I will develop ten "Problem of the Month" problems to be displayed on a bulleting board. I will offer extra credit for the elementary functions course for any student who solves and justifies the problem (problems solutions without justification will receive no credit). Students will have a chance to revise their solutions should they fall short. Students who correctly solve the problem will be recognized by having their names on the "Wall of Fame", as excellent mathematicians who have solved some challenging problem solving problems. It will be my responsibility as the teacher, to make sure that the problems are challenging in nature and that those students who succeed in solving the problems receive proper recognition. Introducing these problems with enthusiasm, and providing consistent reminders about the problem, will be essential in the success of this component.

Problem solving is an excellent source and vehicle for enrichment in mathematics (Posamentier, Smith, & Stepelman, 2006). I used these problems not only as enrichment for my more gifted students, but being that they were simplistic in nature, they were able to be done by all. There was nothing more satisfying than seeing some of my less talented students' sense of satisfaction when they solved one of these problems. Mathematics teachers must always be prepared to provide some appropriate material for the more gifted students, while no neglecting the less-gifted students, who also benefit from appropriately challenging problems (Posamentier, Smith, & Stepelman, 2006).

Several of the problems that I have tried this year with my students, adapted from the Pittsburgh Teachers Institute seminar, have been very successful with students both inside and outside of my class. I would use the problems during testing days or field trips, when several students were absent. I have also

used them in full classes during testing days, when I did not want to bore my students with more mathematics after they had been testing all day. The results were surprising to me. Students were really putting forth their best effort to try and solve the problems. Some students were so excited, that the problem consumed the rest of their day. Although I am sure some of their other teachers were not happy, I was ecstatic to see this level of excitement in mathematics. Some of my students even took the problem to their peers for a friendly challenge, and it resulted in students whom I did not know coming to my classroom to check or get help with a problem. It is this very level of excitement that I want to bring to my classroom on a daily basis.

We also did some activities with the Soma cube, a three dimensional puzzle that students had to put together and record solutions. We made one of these cubes in the seminar, and I brought it to my classroom. I gave a reward of five bonus points for each unique solution (limit 5 solutions). Students of all levels were engaged with this task. I wish I had more Soma cubes in my room, because students began to argue over who got to work with the cube when there was extra time. I had several students come to school early, not to get help with their homework or eat breakfast, but to find more solutions to this puzzle. It consumed several of my students, and led me to wonder how I can further harness this energy in my own classroom.

This Pittsburgh Teachers Institute seminar has helped me discover the power of problem solving in the mathematics classroom, and the interest in mathematics that these types of challenging problems can generate. I believe that if I can use this power more directly with my curriculum, that the possibilities are endless. I believe that if students discover and reach a deeper understanding of the topics of counting, combinations, permutations, and probability, it will make mathematics more fun and applicable to their own lives. I will try to accomplish this by using real world examples. Using such examples as gambling, games of chance, and other real world data, I strongly believe that this will generate a higher interest in mathematics for the students that I teach. I also believe that students, with the help of the teacher, will see the connection between solving the “Problem of the Month” and some of the problems that we solve everyday in class. In fact, it is my hope that by trying to solve some of the more challenging problems that students will find the problems that we solve in class easier and less “boring”.

## **Objectives**

### Section 9.1 – Basic Combinatorics

Students will be able to use the multiplication principle of counting, permutations, or combinations to count the number of ways that a task can be done. Students will explore in pairs or small groups the topics listed above. They will answer sets of questions adapted from *Mathematics: A Human Endeavor*. There are three problems sets for each lesson. All students will complete problem sets one and two for each topic. Problem set three will be enrichment for students who finish the task early, or wish to challenge themselves for extra credit. There are four lessons in this section which must be completed by the student. Any problems not finished must be completed as homework. This unit should take five days. At the end of the section, students will be assigned problems from the textbook, to assess their understanding of the desired objectives. The student will also be assessed on a chapter test at the end of chapter 9.

### Section 9.3 – Probability

Students will be able to identify a sample space and calculate probabilities and conditional probabilities in sample spaces with equally likely or unequally likely outcomes. They will answer sets of questions adapted from *Mathematics: A Human Endeavor*. There are three problems sets for each lesson. All students will complete problem sets one and two for each topic. Problem set three will be enrichment for students who finish the task early, or wish to challenge themselves for extra credit. There are three lessons in this section which must be completed by the student. Any problems not finished must be completed as homework. This unit should take five days. At the end of the section, students will be assigned problems from the textbook, to assess their understanding of the desired objectives. The student will also be assessed on a chapter test at the end of chapter 9.

### Problem of the Month Problems

The objective here is to simply generate interest in the area of problems solving for students. I want all students to be ready, willing, and able to attack the problem that I pick for the problem of the month. I will make reference to the problem every day, so that students do not forget about the problem or think it is of little importance. The problems will not necessarily be related to the elementary functions curriculum, but they will be more in the form of brain teasers, such that any student will have a chance to solve them. A student will present the solution to the class at the end of the month, once the problem has expired. The main goal here is to have students work hard, over a long period of time, to solve a problem that will cause them to think outside the box, and challenge their problem solving skills.

### **Strategies**

## Counting, Permutations, Combinations, and Probability

One of the main strategies that I will use for instruction is group work followed by classroom presentation. As I have stated above, students typically participate in class by answering questions that I ask. They also take notes that I give them, and do their assigned problems. It is my hope that students will generate and answer each other questions as part of these explorations. Perhaps, students will even generate their own problems. Students will read the section in the book and complete problem set one, and we will have a discussion on those results in class. With the remainder of the time, they will work on problem sets two and three, finishing whatever problems they do not complete as homework. The following day, the assignment will be collected and checked for immediate feedback. As stated above, problem set three will be an extra credit assignment, as they seem to be more difficult in nature. While they are working, I will be walking around the room to make sure that each group is on task, as time is definitely a factor.

### Problem of the Month

I will display the problem of the month in an exciting and inviting format, on a bulletin board outside my classroom. It is my hope that students outside of my class may even rise to the challenge that I will present. I need to be consistent with my praise for students who are trying or who have solved the problems, so that other students can see that this is a desirable behavior. I also need to promote the problem, and generate interest in solving the problem. If bonus points do not serve as a proper motivator, then I could give a prize to the first student who solves it correctly, or some other type of award. My consistency in how I present and approach the problem will be the key to its success. Of course, I must also pick challenging and interesting problems that students will enjoy trying to solve. The problems listed below are the problems that I am going to use for the first year of this unit. I will have to change the problems that I pick for subsequent years, so that students do not see repeats.

### **Classroom Activities**

#### Section 9.1 – Basic Combinatorics

##### *Day 1 – Introduction to the counting principle*

Students will be introduced to the topic by a classic problem of an ice cream store that sells two drinks (sodas and milkshakes), in three sizes (small, medium, and large), and four flavors (vanilla, chocolate, strawberry, and cherry). They will be asked to find out how many choices of different drinks that they have. They will

be shown a tree diagram of the situation, and be introduced to the counting principle through the diagram.

The students will then complete problem set one, in pairs or small groups, to various contexts applying the counting principle, and we will discuss our results before the end of the class period. They will then be assigned problem sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs.402-412).

### *Day 2 – Permutations*

Students will be introduced to the topic by a problem in which eight horses are entered in a race and bets are placed on which horse will finish first, second, and third. They will then try to figure out how different ways the first three horses can finish. Students will discuss how to solve this problem using the counting principle. Students will be show a tree diagram modeling the situation, so that they can see how many different possibilities there are.

The students will then complete problem set one, in pairs or small groups, to various contexts applying permutations, and we will discuss our results before the end of the class period. They will then be assigned problem sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs. 413-419).

### *Day 3 – More on Permutations*

Students will be introduced to the topic by a problem where they have to figure out how many different ways the letter of a word can be rearranged that has some repeat letters. Through some exploration, they will be lead to a generalized formula which they can apply to different contexts. They will also see a list of the ways the word could be rearranged, to further show the power of using permutations.

The students will then complete problem set one, in pairs or small groups, to various contexts applying permutations, and we will discuss our results before the end of the class period. They will then be assigned problem sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs. 420-427).

### *Day 4 – Combinations*

Students will be introduced to a topic through a problem asking if a bowler rolls the first ball and knocks over all of the pins but three, how many ways can this

happen? They will realize that it is not the same as the permutations, because the order of the three pins does not matter. They will then develop the formula for combinations and apply it to the problems.

The students will then complete problem set one, in pairs or small groups, to various contexts applying combinations, and we will discuss our results before the end of the class period. They will then be assigned problem sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs. 428-434).

#### *Day 5 – Bringing it all together*

On this day, students will complete the assignment from their regular textbook, where they will apply combinations, permutations, and the counting principle in various contexts. The students will complete problems one through thirty-two on page 675 of their textbook (*Precalculus: Graphical, Numerical, Algebraic*). The assignment will be collected and graded.

### Section 9.3 – Probability

#### *Day 1 – Probability: The Measure of Chance*

Students will be introduced to probability through an example of a Massachusetts Institute of Technology professor and his writings on the game of blackjack. Students will then discover the definition of probability, and realize the range of answers for probability questions.

The students will then complete problem set one, in pairs or small groups, to various contexts applying probability, and we will discuss our results before the end of the class period. They will then be assigned problem sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs. 448-456).

#### *Day 2 – Probabilities of Successive Events*

Students will be introduced to the concept by a problem where they are asked to find the probability of being dealt three cards, with all of them being aces. Students will use combinations to calculate the sample space. Then, students will be shown examples of dependent and independent events.

The students will then complete problem set one, in pairs or small groups, to various contexts applying probability of successive events, and we will discuss our results before the end of the class period. They will then be assigned problem

sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs. 467-475).

### *Day 3 – The Birthday Problem: Complimentary Events*

Students will be introduced to the topic by trying to guess the number of people needed in a group in order for the odds to favor two of them sharing the same birthday. Once they find out the answer, they will discuss complimentary events, and discover that the probabilities of complimentary events always add up to 1.

The students will then complete problem set one, in pairs or small groups, to various contexts applying probability of complimentary events, and we will discuss our results before the end of the class period. They will then be assigned problem sets two and three for homework, which will be collected the following day. (Adapted from *Mathematics: A Human Endeavor*, pgs. 495-503).

### *Day 4 – Conditional Probability*

Using a tree diagram and a problem involving the probability of picking a chocolate chip cookie from a certain jar, students will be introduced to the topic of conditional probability. Students will then develop the conditional probability formula, and apply it to various contexts. We will also review the concept of and vs. or in applying probability.

As a class, we will complete problems one through twenty on page 693 of the textbook (*Precalculus: Graphical, Numerical, Algebraic*), being sure to fill in any gaps caused by using the supplementary materials. The assignment will be collected and graded.

### *Day 5 – Bringing it all together*

On this day, students will complete the assignment from their regular textbook, where they will apply probability in various contexts. The students will complete problems twenty one through fifty on page 693 of their textbook (*Precalculus: Graphical, Numerical, Algebraic*). The assignment will be collected and graded.

### Problems of the Month

*September – FORTY+TEN+TEN=SIXTY*

Each of the letters in the equation below represents one of the digits from 0 to 9. Determine which digit corresponds to each letter.

(<http://ken.duisenberg.com/potw/>)

F O R T Y + T E N + T E N = S I X T Y

I chose this problem because it was similar to some of the ones we did in class. Not only is it a true verbal statement, but it also has a solution using the single digits. It is quite a difficult problem, but most students will use trial and error until they start to figure out some patterns. This is a problem where the math is not difficult, but to solve the problem can be very time consuming.

*October – Friends, Wives, and Money*

Lucky Jim won \$1,000,000 (a million dollars), split it up and gave it all to three male friends and their wives. The wives (together) received just \$4,000 short of \$400,000. Jane got \$10,000 more than Catherine, and Marcy got that same amount more than Jane. John Green was given as much as his wife, Henry Brown got one and a half times as much as his wife received, and Tom Cobalt received twice as much as his. What was the first name of each man's wife, and how much money did each of the six receive? (<http://www.dansmath.com/probofwk>)

I first chose this problem because it deals with money. The solution to find how much each of the wives has is not too difficult algebraically. After students find the wives salary, they must use logic to figure out which husband goes with which wife.

*November – Turk-o-nacci Sequence!*

Like the Fibonacci sequence 1, 1, 2, 3, 5, 8, . . . a certain turkey flock has as many turkeys on a given day as the sum of the number of turkeys on the previous two days. If there were 79 turkeys on November 7th, and 542 turkeys on November 11th, how many turkeys were there on November 18th? (<http://www.dansmath.com/probofwk>)

I chose this problem because of the seasonal application. The level of math involved would be creating and solving a system of equations, although most of my students would use trial and error because they are not familiar with the formulas for the nth term of a Fibonacci sequence. The algebraic solution and proof is quite elegant, and students will be surprised at the results.

*December – Olympic Ring Addition*

Given the five Olympic rings, how can the digits one through nine be placed within the nine regions (five non-overlapping ring regions and four overlapping

regions shared between two rings) so that each ring contains the same total?  
(<http://ken.duisenberg.com/potw/>)

Again, this is another problem where the math is not that difficult, but students will be challenged nonetheless. This puzzle has at least four solutions that I know of, and I will be interested to see how many I get. The algebraic proof for this particular problem is quite difficult, and I do not expect that any of my students will reach such a solution.

### *January – Clinking Glasses*

When I have wine with a few people and we clink glasses and say "salud", I can always tell if everyone has "clinked" with everyone else, because I know math! Let's assume each person clinks each other person exactly once. If there are 2 people, there is one "clink." If there are 3 people, there are 3 clinks.

- How many clinks are there for 4, 5, 6, . . . 10 people?
- How many people were there if I heard 903 clinks?
- What is the formula for the quantity :  $c(n)$  = number of clinks for a group of  $n$  people ? (<http://www.dansmath.com/probofwk>)

This problem uses the counting principle and combinations, a topic that we will be investigating around this time of the year. The problem can be done without counting, but is much more difficult. The formula for part c is able to be found through the use of combinations and the binomial theorem, and must be found first in order to find the answer to part b, unless students use an alternative method.

### *February – Pairs of Socks*

A sock drawer contains some number of red and blue socks (and no other color.) In each case below, find the number of socks of each color in the drawer. A pair of socks is removed at random.

- The probability they are both red is  $1/3$ , and the probability they are both blue is  $1/6$ .
- The probability they are both red is  $1/2$ , and the probability they are both blue is  $1/12$ .
- The probability they are both red is  $1/2$ , and the probability they are both blue is  $1/14$ . (<http://ken.duisenberg.com/potw/>)

This is another problem on probability, which is a topic we will be finishing at the end of the semester, which ends in February. I like this problem because they are given the probabilities for certain situations, and have to work

backwards. The algebraic solution is not impossible, but again, it is my assumption that most students will use trial and error.

### *March – A Sequential Magic Square*

Place nine different digits (from 0-9) into a 3x3 grid, such that the eight sums formed (three rows, three columns, two diagonals) can be arranged as eight sequential numbers. There are two distinct solutions to this problem.

(<http://ken.duisenberg.com/potw/>).

This is another problem that has multiple solutions, and both solutions are related. I believe the students will have trouble coming up with the magic squares, much less ones that involve eight consecutive sums.

### *April – Three Fractions*

Each of three fractions has a one-digit numerator and a two-digit denominator. The three fractions together add up to one. Place the nine digits 1-9 into the fractions to make the equation correct the following equation correct:  $a/bc + d/ef + g/hi = 1$  (<http://ken.duisenberg.com/potw/>)

I picked this problem because it involves fractions, and students typically avoid problems with fractions in it. This problem would not be too difficult for the determined student who has the use of a calculator. Again, the math is not difficult, most students will try different combinations until they find one that works.

### *May – Paint My House*

Aaron, Brenda, Charlie, and Danita are house painters. The team of Aaron, Brenda, and Charlie can paint my house in 3 days; if Aaron and Danita work as a pair it takes 6 days; the pair of Charlie and Danita takes 8 days, and the team of Brenda, Charlie, and Danita can finish in 5 days. How long would it take the whole 4-person crew to paint my house? Answer to the nearest minute.

(<http://www.dansmath.com/probofwk>)

I like this problem because they have to find the time to the nearest minute, using several ratios and proportions to find the correct solution. The algebra is not too difficult, and students may try this approach when solving the problem.

### *June – Train versus Fly*

Two trains are 2 miles apart and are traveling towards each other on the same track, each train going 30 mph. A fly going 60 mph starts at the nose of one train, flies toward the other train, and upon reaching the second train immediately turns around and flies back towards the first train. The fly buzzes back and forth until all three collide. How far did the fly fly? (<http://www.dansmath.com/probofwk>)

This is the last of the problems of the month, and probably the least difficult if the students can figure out how long it takes the two trains to collide. June is a short month, so I chose a problem that is not lengthy. The algebraic solution is fairly simple, and I believe most students will use algebraic methods when solving the problem.

### **Annotated Bibliography/Resources**

Bach, Daniel. (2008). *dansmath.com*. Retrieved May 15, 2008 from <http://www.dansmath.com>

This website was used for several problems of the week and their solutions.

Demana, F., Waits, B.K., Foley, G.D., & Kennedy, D.(Eds.). (2001). *Precalculus: Graphing, numerical, algebraic* (5<sup>th</sup> ed.). Reading, MA. Addison Wesley Longman.

This is the textbook that we use in Elementary Functions, and some of the pages are included below in the appendix.

Duisenberg, Ken. (2008). *ken.duisenberg.com/potw*. Retrieved May 15, 2008 from <http://ken.duisenberg.com/potw/>

This website was used for several problems of the week and their solutions.

Huff, D. & Geis, I. (1959). *How to Take a Chance*. W.W. Norton and Co.

This source has some supplemental and additional problems, including several puzzles and real-life applications.

Jacobs, H. R. (1994). *Mathematics: A Human Endeavor* (3<sup>rd</sup> ed.). W.H. Freeman and Co.

This source was used as the primary source for the alternative teaching method for counting, permutations, combinations, and probability. This source

also contains a number of other activities appropriate for students at this level on varying topics. The pages referenced are included below in the appendix.

McGervey, J. D. (1986). *Probabilities in Every Day Life*. Ivy Books.

This source helps students to learn how to calculate risks that they will take in the real world, and has some strong real world connections such as games, gambling, and insurance. This book was used as a supplementary source to the probability section of this unit.

National Council of Teachers of Mathematics (2000). *Principle and Standards for School Mathematics*. Reston, Virginia. National Council of Teachers of Mathematics.

This source is referenced for the standards listed below and in the article. This is the driving force of mathematics instruction in today's mathematics classroom.

Packel, E. (1981). *The Mathematics of Games and Gambling*. The Mathematical Association of America.

This is another source used for further exploring games of chance and their probabilities. This source was used as supplementary material for the probability section of this unit.

Posamentier, A. S., Smith, B. S., & Stepelman, J. (2006). *Teaching Secondary Mathematics (7<sup>th</sup> Ed.)*. Pearson.

This is an article on the power of problem solving, with several problems included. The article details several different strategies and methods in the area of problem solving, with suggestions for implementation.

## **Appendix-Content Standards**

Pennsylvania State Math Standards

2.1. Numbers, Number Systems and Number Relationships (Addressed in this unit)

2.2. Computation and Estimation

- 2.3. Measurement and Estimation
- 2.4. Mathematical Reasoning and Connections (Addressed in this unit)
- 2.5. Mathematical Problem Solving and Communication (Addressed in this unit)
- 2.6. Statistics and Data Analysis
- 2.7. Probability and Predictions (Addressed in this unit)
- 2.8. Algebra and Functions (Addressed in this unit)
- 2.9. Geometry
- 2.10. Trigonometry
- 2.11. Concepts of Calculus

Problem of the Month Solutions

*September – FORTY+TEN+TEN=SIXTY*

Each of the letters in the equation below represents one of the digits from 0 to 9. Determine which digit corresponds to each letter.

*Solution*

$$29786 + 850 + 850 = 31486$$

*October – Friends, Wives, and Money*

Lucky Jim won \$1,000,000 (a million dollars), split it up and gave it all to three male friends and their wives. The wives (together) received just \$4,000 short of \$400,000. Jane got \$10,000 more than Catherine, and Marcy got that same amount more than Jane. John Green was given as much as his wife, Henry Brown got one and a half times as much as his wife received, and Tom Cobalt received twice as much as his. What was the first name of each man's wife, and how much money did each of the six receive?

*Solution*

We can determine the wives' amounts  $C+M+J=396$  (thousand);  $J=C+10$ ;  $M=J+10$   
 So  $C+(C+10)+(C+10+10)=396$  or  $3*C = 366$   $C=122$ ,  $J=132$ ,  $M=142$ .

At this point, the problem is very finite. The husbands need to add up to 604 (thousand). Rather than just doing cases, let's observe that theoretically pairing any guy with any wife gives an even amount except Henry. The only way to take  $1.5 \times (\text{Wife amount})$  to get an even number is with Jane. So Henry must be married to Jane. Henry gets 198. So either  $2 \times 122 + 142 = 406$  or  $122 + 2 \times 142 = 406$ . It is the latter. So Tom Cobalt (\$284,000) is married to Marcy (\$142,000), John Green (\$122,000) is married to Catherine (\$122,000), and Henry Brown (\$198,000) is married to Jane (\$132,000).

### *November – Turk-o-nacci Sequence!*

Like the Fibonacci sequence 1, 1, 2, 3, 5, 8, . . . a certain turkey flock has as many turkeys on a given day as the sum of the number of turkeys on the previous two days. If there were 79 turkeys on November 7th, and 542 turkeys on November 11th, how many turkeys were there on November 18th?

### *Solution*

Try to work with two unknowns, say the  $A = 5\text{th}$  number and  $B = 6\text{th}$ . Then  $A + B = 79$ .  $B + 79 = B + A + B = A + 2B = 8\text{th}$ ,  $2A + 3B = 9\text{th}$ ,  $3A + 5B = 10\text{th}$ , and  $5A + 8B = 11\text{th} = 542$ . So solving the system  $\{A + B = 79, 5A + 8B = 542\}$  gives  $5A + 5B = 395$ ,  $3B = 147$ ,  $B = 49$ , then  $A = 30$ ; we get 30, 49, **79**, 128, 207, 335, **542**, 877, 1419, 2296, 3715, 6011, 9726, **15737**. **There were 15737 turkeys on the 18th.**

### *December – Olympic Ring Addition*

Given the five Olympic rings, how can the digits one through nine be placed within the nine regions (five non-overlapping ring regions and four overlapping regions shared between two rings) so that each ring contains the same total?

### *Solution*

Label the regions so that  $A+B = B+C+D = D+E+F = F+G+H = H+I$

$$9+2 \quad 2+5+4 \quad 4+6+1 \quad 1+7+3 \quad 3+8 \quad \text{Sum}=11$$

$$9+4 \quad 4+1+8 \quad 8+3+2 \quad 2+5+6 \quad 6+7 \quad \text{Sum}=13$$

$$7+6 \quad 6+5+2 \quad 2+8+3 \quad 3+1+9 \quad 9+4 \quad \text{Sum}=13$$

$$8+6 \quad 6+1+7 \quad 7+4+3 \quad 3+2+9 \quad 9+5 \quad \text{Sum}=14$$

### *January – Clinking Glasses*

When I have wine with a few people and we clink glasses and say "salud", I can always tell if everyone has "clinked" with everyone else, because I know math! Let's assume each person clinks each other person exactly once. If there are 2 people, there is one "clink." If there are 3 people, there are 3 clinks.

- How many clinks are there for 4, 5, 6, . . . 10 people?
- How many people were there if I heard 903 clinks?
- What is the formula for the quantity :  $c(n)$  = number of clinks for a group of  $n$  people ?

### *Solution*

8a. **Answer:** Each person clinks with all others, so for 4 people it's  $3 + 2 + 1 = 6$  clinks. For 5 it's  $4 + 3 + 2 + 1 = 10$ , and we can use the **binomial coefficient**  $nC2$  to do each one. People: Clinks: **4:6, 5:10, 6:15, 7:21, 8:28, 9:36, 10:45.**

8b. **Answer:** We want  $nC2 = 903$  clinks, let's answer (c) first and come back!

8c. **Answer:**  $c(n) = nC2 = n(n-1)/2 = (n^2 - n) / 2$  is "n choose 2", which can be gotten from Pascal's Triangle or from the arithmetic series  $n-1 + n-2 + \dots + 3 + 2 + 1 = n(n-1)/2$  [ there are  $n/2$  pairs that add to  $n-1$  ]. Now to finish #8b...  $nC2 = n(n-1)/2 = 903$ ,  $(n^2 - n) = 1806$ ,  $n^2 - n - 1806 = 0$ ,  $(n - 43)(n + 42) = 0$ ,  $n = 43$  or  $-42$ , so there were **43 people** at the clink-fest.

### *February – Pairs of Socks*

A sock drawer contains some number of red and blue socks (and no other color.) In each case below, find the number of socks of each color in the drawer. A pair of socks is removed at random.

- The probability they are both red is  $1/3$ , and the probability they are both blue is  $1/6$ .
- The probability they are both red is  $1/2$ , and the probability they are both blue is  $1/12$ .
- The probability they are both red is  $1/2$ , and the probability they are both blue is  $1/14$ .

### *Solution*

- 21 red, 15 blue
- 85 red, 35 blue
- 15 red, 6 blue

### *March – A Sequential Magic Square*

Place nine different digits (from 0-9) into a 3x3 grid, such that the eight sums formed (three rows, three columns, two diagonals) can be arranged as eight sequential numbers. There are two distinct solutions to this problem.

*Solution*

Sums from 11 to 18:

1	8	6
9	7	0
4	3	5

Sums from 9 to 16:

3	1	8
9	2	0
4	6	5

*April – Three Fractions*

Each of three fractions has a one-digit numerator and a two-digit denominator. The three fractions together add up to one. Place the nine digits 1-9 into the fractions to make the equation correct the following equation correct:  $a/bc + d/ef + g/hi = 1$

*Solution*

$$5/34 + 7/68 + 9/12 = 1$$

*May – Paint My House*

Aaron, Brenda, Charlie, and Danita are house painters. The team of Aaron, Brenda, and Charlie can paint my house in 3 days; if Aaron and Danita work as a pair it takes 6 days; the pair of Charlie and Danita takes 8 days, and the team of Brenda, Charlie, and Danita can finish in 5 days. How long would it take the whole 4-person crew to paint my house? Answer to the nearest minute.

*Solution*

Let A = days Aaron would take to paint the house himself, B = days for Brenda alone, etc. Then Aaron can do  $1/A$  of the house in a day. If Aaron, Brenda, and Charlie take 3 days, then the fractions for one day give :  $(1/A) + (1/B) + (1/C) = 1/3$  ; letting small letters a, b, c, d replace  $1/A, 1/B, 1/C, 1/D$  : we get the linear system  $\{a + b + c = 1/3, a + d = 1/6 ; c + d = 1/8 ; b + c + d = 1/5\}$ . The solution is:  $a = 3/20, b = 3/40, c = 13/120, d = 1/60$  ; so that all four together would paint

$a + b + c + d = 7/20$  of the house in a day; it would take  $20 / 7$  days which is 2 and  $6/7$  days. Ask Beth and she'll tell you this is **2 days, 20 hours, 34 minutes** and change.

*June – Train versus Fly*

Two trains are 2 miles apart and are traveling towards each other on the same track, each train going 30 mph. A fly going 60 mph starts at the nose of one train, flies toward the other train, and upon reaching the second train immediately turns around and flies back towards the first train. The fly buzzes back and forth until all three collide. How far did the fly fly?

The trains will each go 1 mile in  $1/30$  hr = 2 min, so they collide in 2 min.  
The fly flies 60 mi/hr = 1 mi / min, so in 2 min it goes 2 miles.